Exercise 2.3.4

(The Allee effect) For certain species of organisms, the effective growth rate $\dot{N}/N$ is highest at intermediate N. This is called the Allee effect (Edelstein–Keshet 1988). For example, imagine that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.

a) Show that $\dot{N}/N = r - a(N-b)^2$ provides an example of the Allee effect, if r, a, and b satisfy certain constraints, to be determined.

b) Find all the fixed points of the system and classify their stability.

c) Sketch the solutions $N(t)$ for different initial conditions.

d) Compare the solutions $N(t)$ to those found for the logistic equation. What are the qualitative differences, if any?

Solution

When $N = b$ and $a > 0$, $\dot{N}/N$ is a maximum.

$$\dot{N} = N[r - a(N-b)^2]$$

The fixed points occur when $\dot{N} = 0$.

$$N^*[r - a(N^* - b)^2] = 0$$

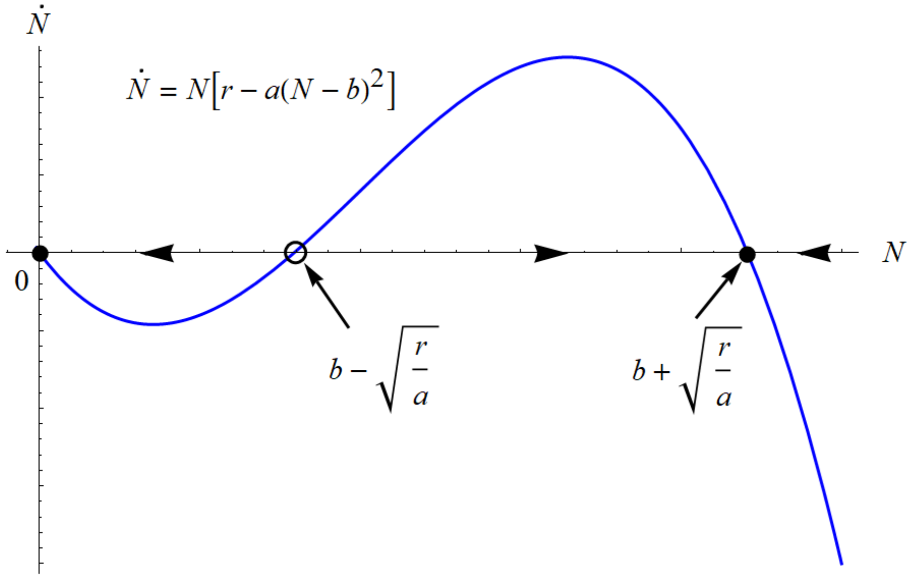
$$N^* = 0 \quad \text{or} \quad r - a(N^* - b)^2 = 0$$

$$N^* = 0 \quad \text{or} \quad N^* = b \pm \sqrt{\frac{r}{a}}$$

In order for there to be a second fixed point, it's necessary to have $r > 0$ (so that the square root yields a real number) and $b > 0$ (so that the population is positive). And for there to be a third fixed point, it's necessary to have

$$b - \sqrt{\frac{r}{a}} > 0.$$

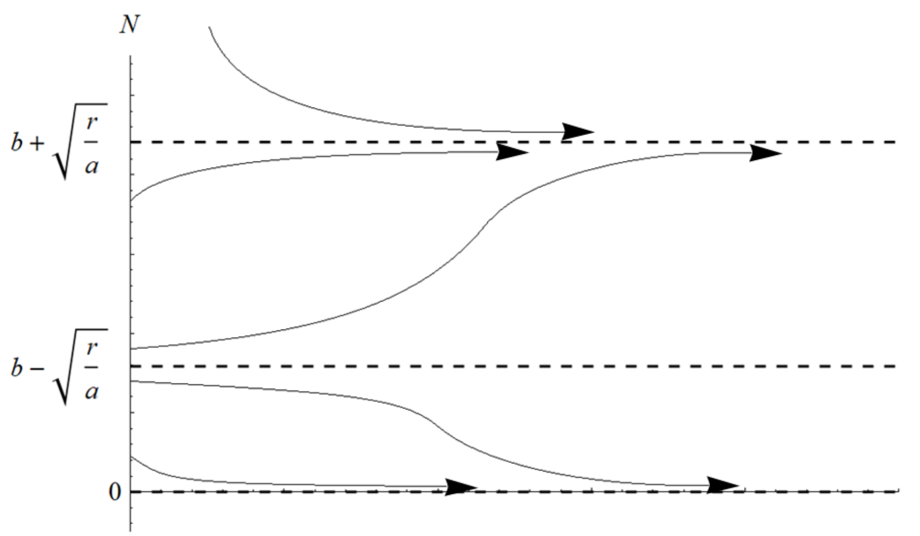
Plot $\dot{N}$ versus N in order to determine the stability of these three points.



When the function is negative the flow is to the left, and when the function is positive the flow is to the right.

$$\begin{cases} N^* = 0 \text{ is a stable fixed point.} \\ N^* = b - \sqrt{\dfrac{r}{a}} \text{ is an unstable fixed point.} \\ N^* = b + \sqrt{\dfrac{r}{a}} \text{ is a stable fixed point.} \end{cases}$$

A qualitative sketch of the population is shown for various initial conditions.



On the other hand, the logistic equation is

$$\dot{N} = rN\left(1 - \frac{N}{K}\right).$$

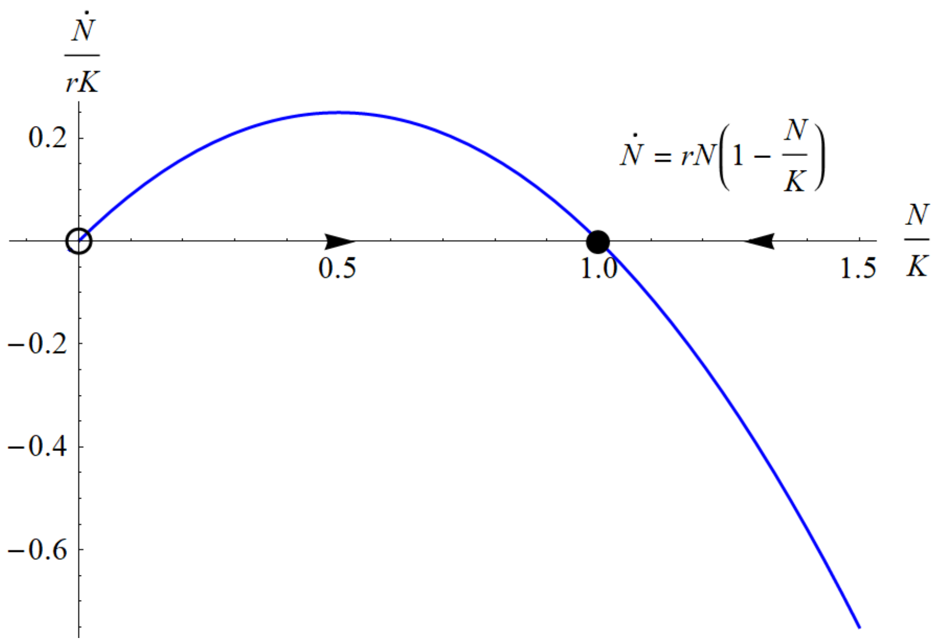
It has fixed points when

$$rN^*\left(1 - \frac{N^*}{K}\right) = 0$$

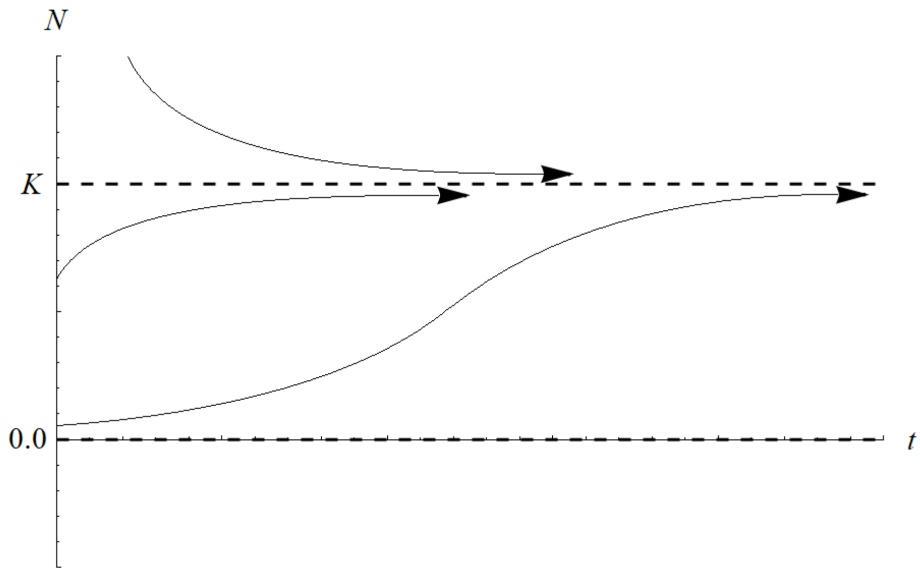
$$rN^* = 0 \quad \text{or} \quad 1 - \frac{N^*}{K} = 0$$

$$N^* = 0 \quad \text{or} \quad N^* = K.$$

Plot $(1/rK)\dot{N}$ versus N/K to determine their stability.



A qualitative sketch of solutions to the logistic equation is shown below for various initial conditions.



Unlike the previous model, the logistic equation fails to account for mating difficulties when N is small.