Exercise 2.4.2

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

$$\dot{x} = x(1-x)(2-x)$$

Solution

The fixed points occur where $\dot{x} = 0$.

$$x^*(1-x^*)(2-x^*) = 0$$

$$x^* = 0 \quad \text{or} \quad 1 - x^* = 0 \quad \text{or} \quad 2 - x^* = 0$$

$$x^* = 0 \quad \text{or} \quad x^* = 1 \quad \text{or} \quad x^* = 2$$

Use linear stability analysis to classify these points.

$$\begin{aligned} f(x) &= x(1-x)(2-x) \\ &= 2x - 3x^2 + x^3 \end{aligned}$$

Differentiate $f(x)$.

$$f'(x) = 2 - 6x + 3x^2$$

As a result,

$$\begin{aligned} f'(0) = 2 > 0 \quad &\Rightarrow \quad x^* = 0 \text{ is an unstable fixed point.} \\ f'(1) = -1 < 0 \quad &\Rightarrow \quad x^* = 1 \text{ is a stable fixed point.} \\ f'(2) = 2 > 0 \quad &\Rightarrow \quad x^* = 2 \text{ is an unstable fixed point.} \end{aligned}$$

The graph of $\dot{x}$ versus x confirms these results.