Exercise 2.4.7

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

$$\dot{x} = ax - x^3, \text{ where } a \text{ can be positive, negative, or zero. Discuss all three cases.}$$

Solution

Case I: $a > 0$

The fixed points occur where $\dot{x} = 0$.

$$ax^* - x^{*3} = 0$$

$$x^*(a - x^{*2}) = 0$$

$$x^*(\sqrt{a} + x^*)(\sqrt{a} - x^*) = 0$$

$$x^* = 0 \quad \text{or} \quad x^* = -\sqrt{a} \quad \text{or} \quad x^* = \sqrt{a}$$

Apply linear stability analysis to determine whether each of these points is stable or unstable.

$$f(x) = ax - x^3$$

Differentiate $f(x)$.

$$f'(x) = a - 3x^2$$

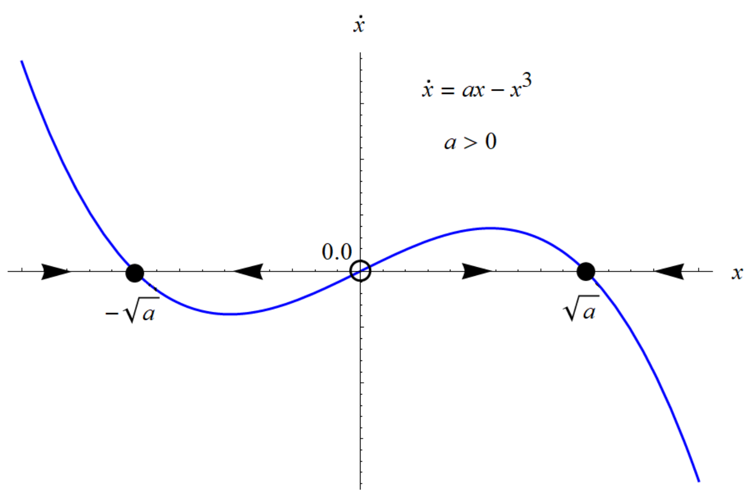
As a result,

$$f'(-\sqrt{a}) = -2a < 0 \quad \Rightarrow \quad x^* = -\sqrt{a} \text{ is a stable fixed point.}$$

$$f'(0) = a > 0 \quad \Rightarrow \quad x^* = 0 \text{ is an unstable fixed point.}$$

$$f'(\sqrt{a}) = -2a < 0 \quad \Rightarrow \quad x^* = \sqrt{a} \text{ is a stable fixed point.}$$

The graph of $\dot{x}$ versus x below confirms these results.



Case II: $a < 0$

The fixed points occur where $\dot{x} = 0$.

$$ax^* - x^{*3} = 0$$

$$x^*(a - x^{*2}) = 0$$

$$x^* = 0$$

Apply linear stability analysis to determine whether this point is stable or unstable.

$$f(x) = ax - x^3$$

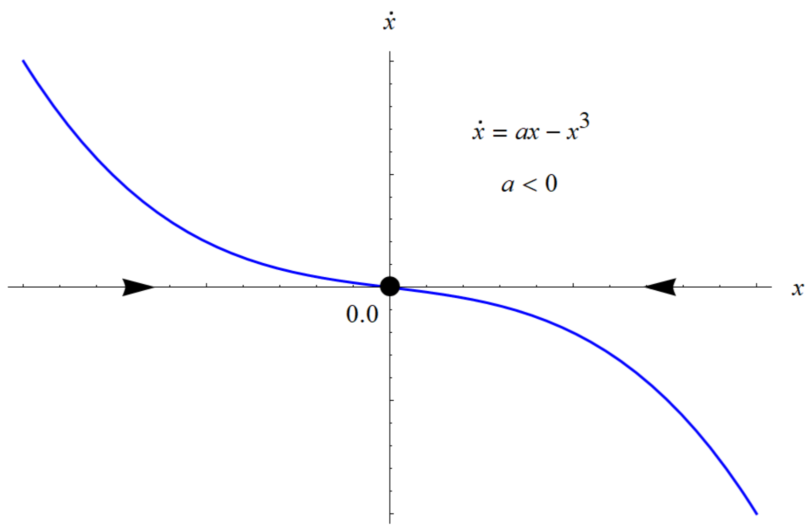
Differentiate $f(x)$.

$$f'(x) = a - 3x^2$$

As a result,

$$f'(0) = a < 0 \qquad \Rightarrow \qquad x^* = 0 \text{ is a stable fixed point.}$$

The graph of $\dot{x}$ versus x below confirms it.



Case III: $a = 0$

The fixed points occur where $\dot{x} = 0$.

$$-x^{*3} = 0$$

$$x^* = 0$$

Apply linear stability analysis to determine whether this point is stable or unstable.

$$f(x) = -x^3$$

Differentiate $f(x)$.

$$f'(x) = -3x^2$$

As a result,

$$f'(0) = 0 \qquad \Rightarrow \qquad \text{No conclusion can be made about the stability of } x^* = 0.$$

The graph of $\dot{x}$ versus x below shows that $x^* = 0$ is a stable fixed point.