Exercise 2.4.9

(Critical slowing down) In statistical mechanics, the phenomenon of "critical slowing down" is a signature of a second-order phase transition. At the transition, the system relaxes to equilibrium much more slowly than usual. Here's a mathematical version of the effect:

a) Obtain the analytical solution to $\dot{x} = -x^3$ for an arbitrary initial condition. Show that $x(t) \to 0$ as $t \to \infty$, but that the decay is not exponential. (You should find that the decay is a much slower algebraic function of t.)

b) To get some intuition about the slowness of the decay, make a numerically accurate plot of the solution for the initial condition $x_0 = 10$, for $0 \leq t \leq 10$. Then, on the same graph, plot the solution to $\dot{x} = -x$ for the same initial condition.

Solution

Part a)

The aim here is to solve the following initial value problem.

$$\frac{dx}{dt} = -x^3, \quad x(0) = x_0$$

Separate variables and then integrate both sides.

$$\frac{dx}{x^3} = -dt$$

$$\int x^{-3}\,dx = \int -dt$$

$$-\frac{x^{-2}}{2} = -t + C \tag{1}$$

Apply the initial condition now to determine C.

$$-\frac{x_0^{-2}}{2} = C$$

Consequently, equation (1) becomes

$$-\frac{x^{-2}}{2} = -t - \frac{x_0^{-2}}{2}$$

$$x^{-2} = 2t + x_0^{-2}$$

$$x^2 = \frac{1}{2t + x_0^{-2}}$$

$$x(t) = \pm\sqrt{\frac{1}{2t + x_0^{-2}}}.$$

The plus sign is chosen so that the right side is $+x_0$ when $t = 0$.

$$x(t) = \frac{1}{\sqrt{2t + x_0^{-2}}}$$

Therefore,

$$x(t) = \frac{x_0}{\sqrt{2x_0^2 t + 1}} \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0.$$

Part b)

On the other hand, the solution to $\dot{x} = -x$ with $x(0) = x_0$ is $x(t) = x_0 e^{-t}$. Below is a plot of the two formulas for $x(t)$ versus t in the special case that $x_0 = 10$.