Exercise 2.5.1

(Reaching a fixed point in a finite time) A particle travels on the half-line $x \geq 0$ with a velocity given by $\dot{x} = -x^c$, where c is real and constant.

a) Find all values of c such that the origin $x = 0$ is a stable fixed point.

b) Now assume that c is chosen such that $x = 0$ is stable. Can the particle ever reach the origin in *a finite* time? Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$, as a function of c?

Solution

Part (a)

Let

$$\dot{x} = -x^c = f(x).$$

If $x = 0$ is a fixed point, then

$$f(0) = 0,$$

which means c cannot be negative or zero: $c > 0$. According to linear stability analysis, in order for $x = 0$ to be a stable fixed point in particular, it's necessary that

$$f'(0) < 0.$$

Differentiate $f(x)$ using the power rule.

$$f'(x) = -cx^{c-1}$$

If $0 < c < 1$, then

$$f'(0) = \lim_{x \to 0^+} \frac{-c}{x^{1-c}} = -\infty.$$

If $c = 1$, then

$$f'(0) = -1.$$

If $c > 1$, then

$$f'(0) = \lim_{x \to 0^+} (-cx^{c-1}) = 0.$$

Therefore, the values of c for which $f'(0) < 0$ is satisfied are $0 < c \leq 1$.

Part (b)

Solve the ODE by separating variables.

$$\dot{x} = -x^c$$

$$\frac{dx}{dt} = -x^c$$

$$x^{-c}\, dx = -dt$$

Integrate both sides.

$$\int x^{-c}\,dx = \int -dt$$

$$\frac{1}{1-c}x^{1-c} = -t + D \tag{1}$$

To determine D, suppose that the particle is at $x = x_0$ when $t = 0$: $x(0) = x_0$.

$$\frac{1}{1-c}x_0^{1-c} = D$$

Consequently, equation (1) becomes

$$\frac{1}{1-c}x^{1-c} = -t + \frac{1}{1-c}x_0^{1-c}.$$

Solve for t.

$$t = \frac{1}{1-c}\left(x_0^{1-c} - x^{1-c}\right).$$

To find how long it takes the particle to reach the origin, set $x = 0$.

$$t = \frac{x_0^{1-c}}{1-c}$$

Provided that $0 < c < 1$, the particle reaches the origin in a finite time. If $x_0 = 1$, then

$$t = \frac{1}{1-c}.$$