

### Exercise 2.5.3

Consider the equation  $\dot{x} = rx + x^3$ , where  $r > 0$  is fixed. Show that  $x(t) \rightarrow \pm\infty$  in finite time, starting from any initial condition  $x_0 \neq 0$ .

#### Solution

Show that  $x(t)$  blows up in finite time by solving the initial value problem.

$$\frac{dx}{dt} = rx + x^3, \quad x(0) = x_0 \neq 0, \quad r > 0$$

Separate variables.

$$\frac{dx}{rx + x^3} = dt$$

Integrate both sides.

$$\int^x \frac{dz}{rz + z^3} = t + C$$

Use partial fraction decomposition on the left side.

$$\int^x \frac{dz}{z(r + z^2)} = t + C$$

$$\int^x \left[ \frac{1}{rz} - \frac{z}{r(r + z^2)} \right] dz = t + C$$

$$\frac{1}{r} \left( \int^x \frac{dz}{z} - \int^x \frac{z}{r + z^2} dz \right) = t + C$$

Make the following substitution in the second integral.

$$u = r + z^2$$

$$du = 2z dz \quad \rightarrow \quad \frac{du}{2} = z dz$$

Consequently,

$$\frac{1}{r} \left[ \ln|x| - \int^{r+x^2} \frac{1}{u} \left( \frac{du}{2} \right) \right] = t + C$$

$$\frac{1}{r} \left( \ln|x| - \frac{1}{2} \int^{r+x^2} \frac{du}{u} \right) = t + C$$

$$\frac{1}{r} \left( \ln|x| - \frac{1}{2} \ln|r + x^2| \right) = t + C$$

$$\frac{1}{r} \left( \ln|x| - \ln\sqrt{r + x^2} \right) = t + C$$

$$\frac{1}{r} \ln \frac{|x|}{\sqrt{r + x^2}} = t + C.$$

Apply the initial condition  $x(0) = x_0$  to determine  $C$ .

$$\frac{1}{r} \ln \frac{|x_0|}{\sqrt{r+x_0^2}} = 0 + C \quad \rightarrow \quad C = \frac{1}{r} \ln \frac{|x_0|}{\sqrt{r+x_0^2}}$$

So the solution becomes

$$\frac{1}{r} \ln \frac{|x|}{\sqrt{r+x^2}} = t + \frac{1}{r} \ln \frac{|x_0|}{\sqrt{r+x_0^2}}$$

$$\frac{1}{r} \left( \ln \frac{|x|}{\sqrt{r+x^2}} - \ln \frac{|x_0|}{\sqrt{r+x_0^2}} \right) = t$$

$$\frac{1}{r} \ln \frac{|x|}{\sqrt{r+x^2}} \frac{\sqrt{r+x_0^2}}{|x_0|} = t$$

$$\ln \frac{|x|}{\sqrt{r+x^2}} \frac{\sqrt{r+x_0^2}}{|x_0|} = rt$$

$$\frac{|x|}{\sqrt{r+x^2}} \frac{\sqrt{r+x_0^2}}{|x_0|} = e^{rt}$$

$$\frac{x^2}{r+x^2} \frac{r+x_0^2}{x_0^2} = e^{2rt}$$

$$\frac{1}{\frac{r}{x^2} + 1} = \frac{x_0^2 e^{2rt}}{r+x_0^2}$$

$$\frac{r}{x^2} + 1 = \frac{r+x_0^2}{x_0^2 e^{2rt}}$$

$$\frac{r}{x^2} = \frac{r+x_0^2}{x_0^2 e^{2rt}} - 1$$

$$\frac{r}{x^2} = \frac{r+x_0^2 - x_0^2 e^{2rt}}{x_0^2 e^{2rt}}$$

$$\frac{x^2}{r} = \frac{x_0^2 e^{2rt}}{r+x_0^2 - x_0^2 e^{2rt}}$$

$$x^2 = \frac{rx_0^2 e^{2rt}}{r+x_0^2(1-e^{2rt})}$$

Therefore, taking the square root of both sides,

$$x(t) = \pm \sqrt{\frac{rx_0^2 e^{2rt}}{r+x_0^2(1-e^{2rt})}}$$

Note that if  $x_0 = 0$ , then  $x(t) = 0$ ; it would take an infinite amount of time for  $x(t)$  to blow up.

If  $x_0 \neq 0$ , then  $x(t)$  blows up when the denominator is zero.

$$r + x_0^2(1 - e^{2rt}) = 0$$

$$x_0^2(1 - e^{2rt}) = -r$$

$$1 - e^{2rt} = -\frac{r}{x_0^2}$$

$$-e^{2rt} = -\frac{r}{x_0^2} - 1$$

$$e^{2rt} = \frac{r}{x_0^2} + 1$$

$$\ln e^{2rt} = \ln\left(\frac{r}{x_0^2} + 1\right)$$

$$2rt = \ln\left(\frac{r}{x_0^2} + 1\right)$$

$$t = \frac{1}{2r} \ln\left(\frac{r}{x_0^2} + 1\right)$$

Therefore,  $x(t) \rightarrow \pm\infty$  in finite time, starting from any initial condition  $x_0 \neq 0$ .