Exercise 2.7.7

(Another proof that solutions to $\dot{x} = f(x)$ can't oscillate) Let $\dot{x} = f(x)$ be a vector field on the line. Use the existence of a potential function $V(x)$ to show that solutions $x(t)$ cannot oscillate.

Solution

Assume the existence of a potential function $V(x)$.

$$\dot{x} = f(x) = -\frac{dV}{dx}$$

Since $x = x(t)$, the chain rule yields

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx}(\dot{x}) = \frac{dV}{dx}\left(-\frac{dV}{dx}\right) = -\left(\frac{dV}{dx}\right)^2 \leq 0,$$

indicating that $V(t)$ never increases along a particle's trajectory. If there is a solution that oscillates with smallest period T,

$$x(t) = x(t+T), \quad T > 0,$$

then the potential must remain constant (it can never increase, so it can never decrease either) at

$$V(x(t)) = V(x(t+T)).$$

But then $dV/dx = 0$, which makes

$$\dot{x} = 0 \quad \rightarrow \quad x(t) = \text{constant}.$$

This is a contradiction because $x(t)$ is supposed to be oscillatory, not constant. The assumption made that an oscillatory solution with period T exists must be false then. Therefore, solutions to $\dot{x} = f(x)$ do not oscillate.