Problem 1.13

Let $\mathbf{u}$ be an arbitrary fixed unit vector and show that any vector $\mathbf{b}$ satisfies

$$b^2 = (\mathbf{u} \cdot \mathbf{b})^2 + (\mathbf{u} \times \mathbf{b})^2.$$

Explain this result in words, with the help of a picture.

Solution

Start with the right side of the identity and show that it simplifies to b^2. Suppose that the angle between $\mathbf{u}$ and $\mathbf{b}$ is θ.

$$\begin{aligned}
(\mathbf{u} \cdot \mathbf{b})^2 + (\mathbf{u} \times \mathbf{b})^2 &= (\mathbf{u} \cdot \mathbf{b})^2 + (\mathbf{u} \times \mathbf{b}) \cdot (\mathbf{u} \times \mathbf{b}) \\
&= (\mathbf{u} \cdot \mathbf{b})^2 + |\mathbf{u} \times \mathbf{b}||\mathbf{u} \times \mathbf{b}| \cos 0 \\
&= (\mathbf{u} \cdot \mathbf{b})^2 + |\mathbf{u} \times \mathbf{b}|^2 \\
&= (|\mathbf{u}||\mathbf{b}| \cos \theta)^2 + (|\mathbf{u}||\mathbf{b}| \sin \theta)^2 \\
&= |\mathbf{u}|^2|\mathbf{b}|^2 \cos^2 \theta + |\mathbf{u}|^2|\mathbf{b}|^2 \sin^2 \theta \\
&= |\mathbf{u}|^2|\mathbf{b}|^2(\cos^2 \theta + \sin^2 \theta) \\
&= |\mathbf{u}|^2|\mathbf{b}|^2 \\
&= u^2 b^2
\end{aligned}$$

Since $\mathbf{u}$ is a unit vector, its magnitude is 1: $u^2 = 1$. Therefore,

$$(\mathbf{u} \cdot \mathbf{b})^2 + (\mathbf{u} \times \mathbf{b})^2 = b^2.$$

The dot product of $\mathbf{b}$ and $\mathbf{u}$ represents the component of $\mathbf{b}$ parallel to the direction of u, and the cross product of $\mathbf{b}$ and $\mathbf{u}$ represents the component of $\mathbf{b}$ perpendicular to the direction of $\mathbf{u}$.

$$b_{\parallel}^2 + b_{\perp}^2 = b^2$$