Problem 1.14

Prove that for any two vectors $\mathbf{a}$ and $\mathbf{b}$,

$$|\mathbf{a}+\mathbf{b}| \le (a+b).$$

[*Hint:* Work out $|\mathbf{a}+\mathbf{b}|^2$ and compare it with $(a+b)^2$.] Explain why this is called the triangle inequality.

Solution

Simplify $|\mathbf{a}+\mathbf{b}|^2$. Suppose that the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ.

$$\begin{aligned}
|\mathbf{a}+\mathbf{b}|^2 &= |\mathbf{a}+\mathbf{b}||\mathbf{a}+\mathbf{b}|\cos 0 \\
&= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) \\
&= \mathbf{a}\cdot(\mathbf{a}+\mathbf{b}) + \mathbf{b}\cdot(\mathbf{a}+\mathbf{b}) \\
&= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} \\
&= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} \\
&= \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} \\
&= |\mathbf{a}||\mathbf{a}|\cos 0 + 2|\mathbf{a}||\mathbf{b}|\cos\theta + |\mathbf{b}||\mathbf{b}|\cos 0 \\
&= |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}|\cos\theta + |\mathbf{b}|^2 \\
&= a^2 + 2ab\cos\theta + b^2
\end{aligned}$$

Note that $(a+b)^2 = a^2 + 2ab + b^2$, so

$$|\mathbf{a}+\mathbf{b}|^2 \le (a+b)^2$$

because $\cos\theta \le 1$. Therefore, taking the square root of both sides,

$$|\mathbf{a}+\mathbf{b}| \le (a+b)$$

This is known as the triangle inequality because it indicates that no side of a triangle is longer than the sum of the other two.