Problem 1.15

Show that the definition (1.9) of the cross product is equivalent to the elementary definition that $\mathbf{r} \times \mathbf{s}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{s}$, with magnitude $rs\sin\theta$ and direction given by the right-hand rule. [*Hint:* It is a fact (though quite hard to prove) that the definition (1.9) is independent of your choice of axes. Therefore you can choose axes so that $\mathbf{r}$ points along the x axis and $\mathbf{s}$ lies in the xy plane.]

Solution

Equation (1.9) in the text defines the cross product in terms of a determinant.

$$\mathbf{r} \times \mathbf{s} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{vmatrix} = (r_y s_z - r_z s_y)\hat{\mathbf{x}} - (r_x s_z - r_z s_x)\hat{\mathbf{y}} + (r_x s_y - r_y s_x)\hat{\mathbf{z}}$$

To show that $\mathbf{r} \times \mathbf{s}$ is perpendicular to $\mathbf{r}$, calculate their dot product and show that it's equal to zero.

$$\begin{aligned} (\mathbf{r} \times \mathbf{s}) \cdot \mathbf{r} &= (r_y s_z - r_z s_y) r_x - (r_x s_z - r_z s_x) r_y + (r_x s_y - r_y s_x) r_z \\ &= \cancel{r_x r_y s_z} - \cancel{r_x r_z s_y} - \cancel{r_x r_y s_z} + \cancel{r_y r_z s_x} + \cancel{r_x r_z s_y} - \cancel{r_y r_z s_x} \\ &= 0 \end{aligned}$$

To show that $\mathbf{r} \times \mathbf{s}$ is perpendicular to $\mathbf{s}$, calculate their dot product and show that it's equal to zero.

$$\begin{aligned} (\mathbf{r} \times \mathbf{s}) \cdot \mathbf{s} &= (r_y s_z - r_z s_y) s_x - (r_x s_z - r_z s_x) s_y + (r_x s_y - r_y s_x) s_z \\ &= \cancel{r_y s_x s_z} - \cancel{r_z s_x s_y} - \cancel{r_x s_y s_z} + \cancel{r_z s_x s_y} + \cancel{r_x s_y s_z} - \cancel{r_y s_x s_z} \\ &= 0 \end{aligned}$$

The aim now is to show that the magnitude of $\mathbf{r} \times \mathbf{s}$ is $rs\sin\theta$, where θ is the angle between $\mathbf{r}$ and $\mathbf{s}$.

$$\begin{aligned} |\mathbf{r} \times \mathbf{s}| &= \sqrt{(r_y s_z - r_z s_y)^2 + [-(r_x s_z - r_z s_x)]^2 + (r_x s_y - r_y s_x)^2} \\ &= \sqrt{r_y^2 s_z^2 + r_z^2 s_y^2 - 2 r_y r_z s_y s_z + r_x^2 s_z^2 + r_z^2 s_x^2 - 2 r_x r_z s_x s_z + r_x^2 s_y^2 + r_y^2 s_x^2 - 2 r_x r_y s_x s_y} \end{aligned} \tag{1}$$

Use the definition of the dot product.

$$\mathbf{r} \cdot \mathbf{s} = rs\cos\theta$$

Solve for $\cos\theta$.

$$\cos\theta = \frac{\mathbf{r} \cdot \mathbf{s}}{rs}$$

Square both sides.

$$\cos^2\theta = \frac{(\mathbf{r} \cdot \mathbf{s})^2}{r^2 s^2}$$

Substitute $\cos^2\theta = 1 - \sin^2\theta$.

$$1 - \sin^2\theta = \frac{(\mathbf{r} \cdot \mathbf{s})^2}{r^2 s^2}$$

Solve for $\sin\theta$.

$$\begin{aligned}\sin\theta &= \pm\sqrt{1-\frac{(\mathbf{r}\cdot\mathbf{s})^2}{r^2s^2}}\\ &= \pm\frac{\sqrt{r^2s^2-(\mathbf{r}\cdot\mathbf{s})^2}}{rs}\end{aligned}$$

Calculate $rs\sin\theta$, choosing the positive root because the magnitude is positive, and show that it gives the same result as equation (1).

$$\begin{aligned}rs\sin\theta &= rs\left(\frac{\sqrt{r^2s^2-(\mathbf{r}\cdot\mathbf{s})^2}}{rs}\right)\\ &= \sqrt{r^2s^2-(\mathbf{r}\cdot\mathbf{s})^2}\\ &= \sqrt{(r_x^2+r_y^2+r_z^2)(s_x^2+s_y^2+s_z^2)-(r_xs_x+r_ys_y+r_zs_z)^2}\\ &= \sqrt{r_y^2s_z^2+r_z^2s_y^2-2r_yr_zs_ys_z+r_x^2s_z^2+r_z^2s_x^2-2r_xr_zs_xs_z+r_x^2s_y^2+r_y^2s_x^2-2r_xr_ys_xs_y}\end{aligned}$$

Therefore, $|\mathbf{r}\times\mathbf{s}| = rs\sin\theta$.