Problem 1.18

The three vectors **a**, **b**, **c** are the three sides of the triangle ABC with angles α, β, γ as shown in Figure 1.15. **(a)** Prove that the area of the triangle is given by any one of these three expressions:

$$\text{area} = \frac{1}{2}|\mathbf{a}\times\mathbf{b}| = \frac{1}{2}|\mathbf{b}\times\mathbf{c}| = \frac{1}{2}|\mathbf{c}\times\mathbf{a}|.$$

(b) Use the equality of these three expressions to prove the so-called law of sines, that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

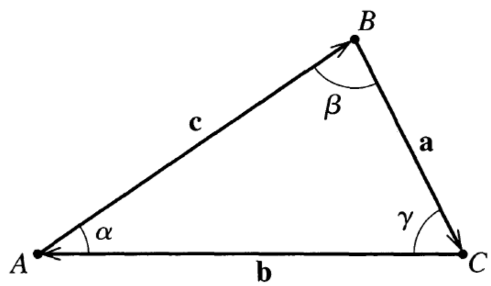


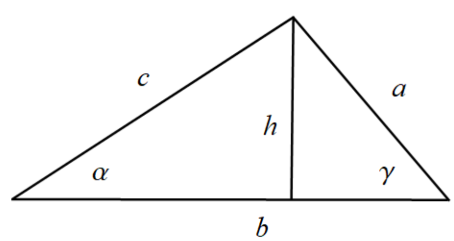
Figure 1.15 Triangle for Problem 1.18.

Solution

The area of a triangle is half the base times the height.

$$A = \frac{1}{2}BH$$

One way to draw the given triangle is with **b** forming the base.



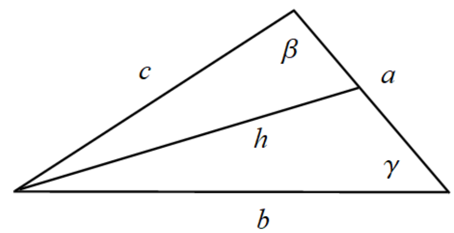
Use trigonometry to determine the height.

$$\sin\alpha = \frac{h}{c} \quad\rightarrow\quad h = c\sin\alpha$$
$$\sin\gamma = \frac{h}{a} \quad\rightarrow\quad h = a\sin\gamma$$

The area is then

$$A = \frac{1}{2}b(c\sin\alpha) = \frac{1}{2}b(a\sin\gamma).$$

A second way to draw the given triangle is with **a** forming the base.



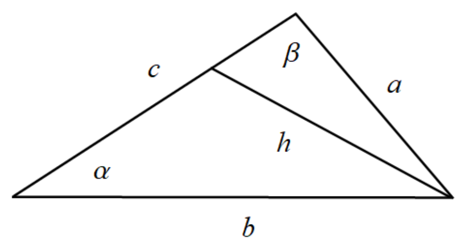
Use trigonometry to determine the height.

$$\sin\beta = \frac{h}{c} \quad \rightarrow \quad h = c\sin\beta$$
$$\sin\gamma = \frac{h}{b} \quad \rightarrow \quad h = b\sin\gamma$$

The area is then also

$$A = \frac{1}{2}a(c\sin\beta) = \frac{1}{2}a(b\sin\gamma).$$

A third way to draw the given triangle is with **c** forming the base.



Use trigonometry to determine the height.

$$\sin\beta = \frac{h}{a} \quad \rightarrow \quad h = a\sin\beta$$
$$\sin\alpha = \frac{h}{b} \quad \rightarrow \quad h = b\sin\alpha$$

The area is then also

$$A = \frac{1}{2}c(a\sin\beta) = \frac{1}{2}c(b\sin\alpha).$$

From all the formulas obtained, there are three unique ways to express the area of this triangle.

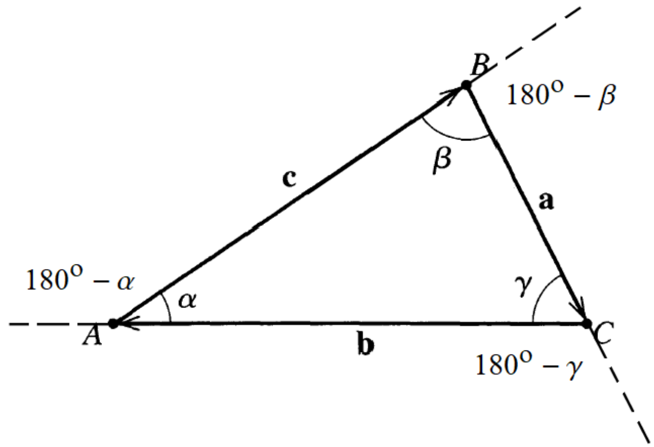
$$A = \frac{1}{2}ab\sin\gamma = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin\beta$$

To get the law of sines, multiply all sides by $2/(abc)$

$$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

and then invert them.

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$



According to this figure, the angle between vectors, **b** and **c**, is $180° - \alpha$; the angle between vectors, **c** and **a**, is $180° - \beta$; and the angle between vectors, **a** and **b**, is $180° - \gamma$. Use the identity $\sin x = \sin(180° - x)$ to rewrite the formulas for the area.

$$A = \frac{1}{2}ab\sin(180° - \gamma) = \frac{1}{2}bc\sin(180° - \alpha) = \frac{1}{2}ca\sin(180° - \beta)$$

Therefore, in terms of the cross product, the area is

$$A = \frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}|\mathbf{b} \times \mathbf{c}| = \frac{1}{2}|\mathbf{c} \times \mathbf{a}|.$$