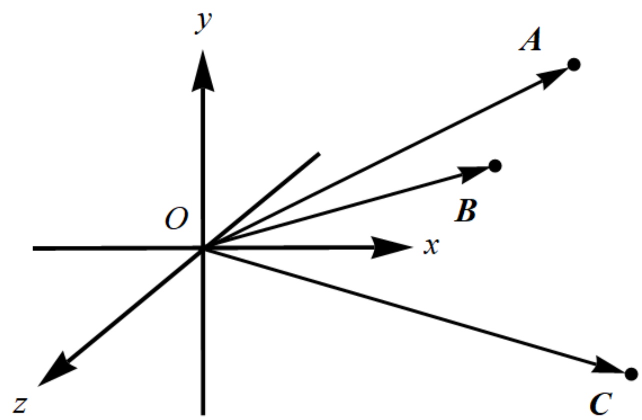
Problem 1.20

The three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ point from the origin O to the three corners of a triangle. Use the result of Problem 1.18 to show that the area of the triangle is given by

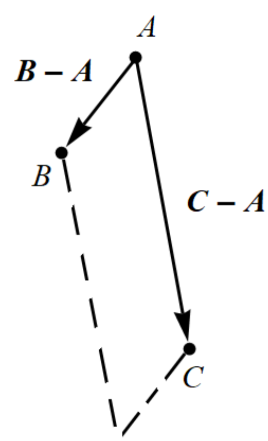
$$\text{(area of triangle)} = \frac{1}{2}\left|(\mathbf{B}\times\mathbf{C}) + (\mathbf{C}\times\mathbf{A}) + (\mathbf{A}\times\mathbf{B})\right|.$$

Solution

Draw the three position vectors, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, to the triangle vertices.



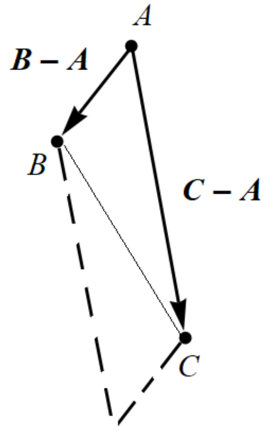
The displacement vector from point A to point B is $\mathbf{B} - \mathbf{A}$, and the displacement vector from point A to point C is $\mathbf{C} - \mathbf{A}$.



The area of the parallelogram formed by these displacement vectors is

$$\text{(area of parallelogram)} = |(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})|,$$

and the desired area of the triangle is half of this.



$$
\begin{aligned}
\text{(area of triangle)} &= \frac{1}{2}|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})| \\
&= \frac{1}{2}|\mathbf{B}\times(\mathbf{C}-\mathbf{A})-\mathbf{A}\times(\mathbf{C}-\mathbf{A})| \\
&= \frac{1}{2}|\mathbf{B}\times\mathbf{C}-\mathbf{B}\times\mathbf{A}-\mathbf{A}\times\mathbf{C}+\underbrace{\mathbf{A}\times\mathbf{A}}_{=\mathbf{0}}| \\
&= \frac{1}{2}|\mathbf{B}\times\mathbf{C}+\mathbf{A}\times\mathbf{B}+\mathbf{C}\times\mathbf{A}|
\end{aligned}
$$