Problem 1.22

The two vectors **a** and **b** lie in the xy plane and make angles α and β with the x axis. **(a)** By evaluating $\mathbf{a}\cdot\mathbf{b}$ in two ways [namely using (1.6) and (1.7)] prove the well-known trig identity

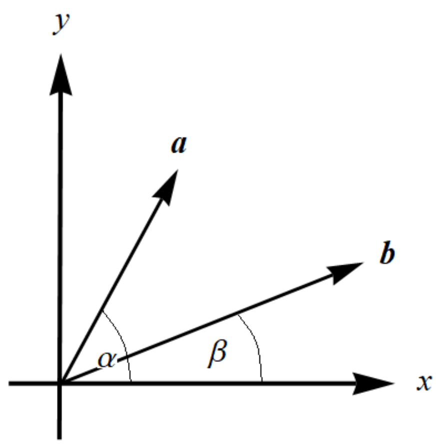
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

(b) By similarly evaluating $\mathbf{a}\times\mathbf{b}$ prove that

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\cos\beta.$$

Solution

The vectors, **a** and **b**, are shown below in the xy-plane.



Consider the dot product and its two formulas. Note that the angle between **a** and **b** is $\alpha-\beta$.

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos(\alpha-\beta)$$
$$\mathbf{a}\cdot\mathbf{b}=a_xb_x+a_yb_y=(|\mathbf{a}|\cos\alpha)(|\mathbf{b}|\cos\beta)+(|\mathbf{a}|\sin\alpha)(|\mathbf{b}|\sin\beta)=|\mathbf{a}||\mathbf{b}|(\cos\alpha\cos\beta+\sin\alpha\sin\beta)$$

Equate these two expressions for $\mathbf{a}\cdot\mathbf{b}$.

$$|\mathbf{a}||\mathbf{b}|\cos(\alpha-\beta)=|\mathbf{a}||\mathbf{b}|(\cos\alpha\cos\beta+\sin\alpha\sin\beta)$$

Therefore, dividing both sides by $|\mathbf{a}||\mathbf{b}|$,

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

Consider the cross product and its two formulas. Note that by the right-hand corkscrew rule, the direction of $\mathbf{a}\times\mathbf{b}$ is into the paper, or the $-\hat{\mathbf{z}}$ direction.

$$\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin(\alpha-\beta)(-\hat{\mathbf{z}})$$
$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{x}}&\hat{\mathbf{y}}&\hat{\mathbf{z}}\\a_x&a_y&a_z\\b_x&b_y&b_z\end{vmatrix}=\begin{vmatrix}\hat{\mathbf{x}}&\hat{\mathbf{y}}&\hat{\mathbf{z}}\\|\mathbf{a}|\cos\alpha&|\mathbf{a}|\sin\alpha&0\\|\mathbf{b}|\cos\beta&|\mathbf{b}|\sin\beta&0\end{vmatrix}=[(|\mathbf{a}|\cos\alpha)(|\mathbf{b}|\sin\beta)-(|\mathbf{a}|\sin\alpha)(|\mathbf{b}|\cos\beta)]\hat{\mathbf{z}}$$

Equate these two expressions for $\mathbf{a}\times\mathbf{b}$.

$$|\mathbf{a}||\mathbf{b}|\sin(\alpha-\beta)(-\hat{\mathbf{z}})=|\mathbf{a}||\mathbf{b}|(\cos\alpha\sin\beta-\sin\alpha\cos\beta)\hat{\mathbf{z}}$$

Dot both sides by $\hat{\mathbf{z}}$.

$$-|\mathbf{a}||\mathbf{b}|\sin(\alpha-\beta) = |\mathbf{a}||\mathbf{b}|(\cos\alpha\sin\beta - \sin\alpha\cos\beta)$$

Therefore, dividing both sides by $-|\mathbf{a}||\mathbf{b}|$,

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$