Problem 1.24

In case you haven't studied any differential equations before, I shall be introducing the necessary ideas as needed. Here is a simple exercise to get you started: Find the general solution of the first-order equation $df/dt = f$ for an unknown function $f(t)$. [There are several ways to do this. One is to rewrite the equation as $df/f = dt$ and then integrate both sides.] How many arbitrary constants does the general solution contain? [Your answer should illustrate the important general theorem that the solution to any nth-order differential equation (in a very large class of "reasonable" equations) contains n arbitrary constants.]

Solution

$$\frac{df}{dt} = f$$

Divide both sides by f.

$$\frac{1}{f}\frac{df}{dt} = 1$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt}(\ln f) = 1$$

Integrate both sides with respect to t.

$$\ln f = t + C$$

Exponentiate both sides.

$$e^{\ln f} = e^{t+C}$$

$$f(t) = e^t e^C$$

Therefore, using a new constant A for e^C,

$$f(t) = Ae^t.$$

There's only one arbitrary constant here because this ODE is first-order. An alternative way to solve the ODE is by separating variables, Mr. Taylor's suggested method.

$$\frac{df}{dt} = f$$

$$\frac{df}{f} = dt$$

Integrate both sides.

$$\int \frac{df}{f} = \int dt$$

$$\ln f = t + C$$

Exponentiate both sides.

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