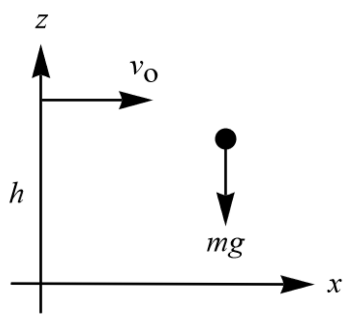
Problem 1.36

A plane, which is flying horizontally at a constant speed v_o and at a height h above the sea, must drop a bundle of supplies to a castaway on a small raft. **(a)** Write down Newton's second law for the bundle as it falls from the plane, assuming you can neglect air resistance. Solve your equations to give the bundle's position in flight as a function of time t. **(b)** How far before the raft (measured horizontally) must the pilot drop the bundle if it is to hit the raft? What is this distance if $v_o = 50$ m/s, $h = 100$ m, and $g \approx 10$ m/s^2? **(c)** Within what interval of time ($\pm\Delta t$) must the pilot drop the bundle if it is to land within ± 10 m of the raft?

Solution

Part (a)

Start by drawing a free-body diagram of the bundle. Because there's no air resistance, there's only a gravitational force acting on the bundle.



Newton's second law states that the sum of the forces on the bundle is equal to its mass times acceleration.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_x = ma_x \\ \sum F_y = ma_y \\ \sum F_z = ma_z \end{cases}$$

The only force is due to gravity, and it's in the negative z-direction.

$$\begin{cases} 0 = ma_x \\ 0 = ma_y \\ -mg = ma_z \end{cases}$$

Divide both sides of each equation by m.

$$\begin{cases} 0 = a_x \\ 0 = a_y \\ -g = a_z \end{cases}$$

Acceleration is the second derivative of position.

$$\begin{cases} \frac{d^2x}{dt^2} = 0 \\ \frac{d^2y}{dt^2} = 0 \\ \frac{d^2z}{dt^2} = -g \end{cases}$$

Integrate both sides of each equation with respect to t to get the components of the bundle's velocity.

$$\begin{cases} \frac{dx}{dt} = C_1 \\ \frac{dy}{dt} = C_2 \\ \frac{dz}{dt} = -gt + C_3 \end{cases} \tag{1}$$

The initial velocity in the x-, y-, and z-directions are v_o, 0, and 0, respectively.

$$\frac{dx}{dt}(0) = C_1 = v_o \quad \rightarrow \quad C_1 = v_o$$

$$\frac{dy}{dt}(0) = C_2 = 0 \quad \rightarrow \quad C_2 = 0$$

$$\frac{dz}{dt}(0) = -g(0) + C_3 = 0 \quad \rightarrow \quad C_3 = 0$$

As a result, equation (1) becomes

$$\begin{cases} \frac{dx}{dt} = v_o \\ \frac{dy}{dt} = 0 \\ \frac{dz}{dt} = -gt \end{cases}.$$

Integrate both sides of each equation with respect to t once more to get the components of the bundle's position.

$$\begin{cases} x(t) = v_o t + C_4 \\ y(t) = C_5 \\ z(t) = -\frac{gt^2}{2} + C_6 \end{cases} \tag{2}$$

The bundle's initial position is $x = 0$, $y = 0$, and $z = h$ when $t = 0$.

$$x(0) = v_o(0) + C_4 = 0 \quad \rightarrow \quad C_4 = 0$$

$$y(0) = C_5 = 0 \quad \rightarrow \quad C_5 = 0$$

$$z(0) = -\frac{g(0)^2}{2} + C_6 = h \quad \rightarrow \quad C_6 = h$$

Consequently, equation (2) becomes

$$\begin{cases} x(t) = v_o t \\ y(t) = 0 \\ z(t) = -\dfrac{gt^2}{2} + h \end{cases}.$$

Therefore, the bundle's position is

$$\boxed{\mathbf{r}(t) = \left\langle v_o t, 0, -\frac{gt^2}{2} + h \right\rangle.}$$

Part (b)

To find how long the bundle is in the air for, set $z(t) = 0$ and solve for nonzero t.

$$z(t) = -\frac{gt^2}{2} + h = 0$$

$$gt^2 = 2h$$

$$t = \sqrt{\frac{2h}{g}}$$

Now plug this nonzero time into $x(t)$ to determine how far the bundle travels while it's in the air.

$$\boxed{x\left(\sqrt{\frac{2h}{g}}\right) = v_o\sqrt{\frac{2h}{g}}}$$

If $v_o = 50$ m/s, $h = 100$ m, and $g \approx 10$ m/s^2, then

$$\boxed{x\left(\sqrt{\frac{2h}{g}}\right) = (50)\sqrt{\frac{2(100)}{10}} \text{ m} = 100\sqrt{5} \text{ m} \approx 224 \text{ m}.}$$

Part (c)

From the x-component of the bundle's position,

$$x(t) = v_o t \quad \Rightarrow \quad \Delta x = v_o \Delta t.$$

Solve for Δt.

$$\Delta t = \frac{\Delta x}{v_o}$$

Therefore, the pilot must drop the bundle within an interval of time,

$$\pm\frac{10 \text{ m}}{v_o},$$

for it to land within ± 10 m of the raft. If $v_o = 50$ m/s, then this time interval is ± 0.2 seconds.