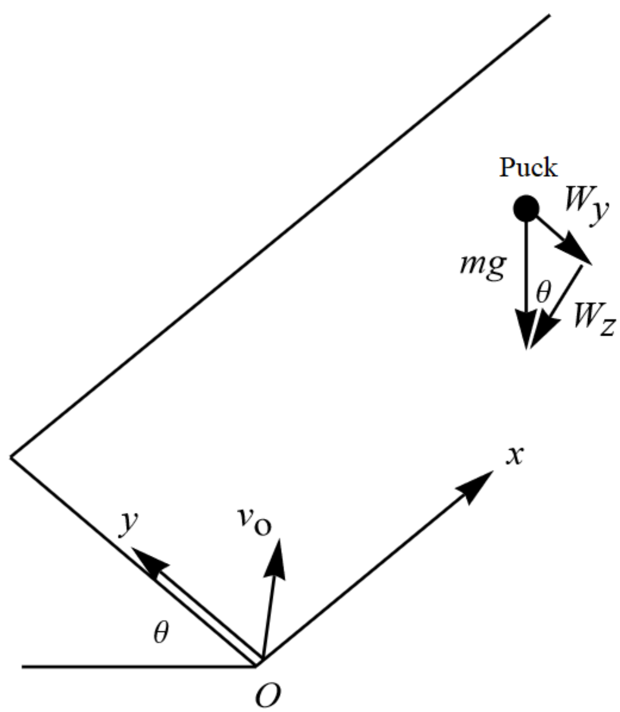
Problem 1.38

You lay a rectangular board on the horizontal floor and then tilt the board about one edge until it slopes at angle θ with the horizontal. Choose your origin at one of the two corners that touch the floor, the x axis pointing along the bottom edge of the board, the y axis pointing up the slope, and the z axis normal to the board. You now kick a frictionless puck that is resting at O so that it slides across the board with initial velocity $(v_{\text{o}x}, v_{\text{o}y}, 0)$. Write down Newton's second law using the given coordinates and then find how long the puck takes to return to the floor level and how far it is from O when it does so.

Solution

Start by drawing the free-body diagram for the puck. Note that because the puck is frictionless, there's only the force of gravity acting on it.



W_y and W_z are the components (their magnitudes, rather) of the weight vector along the y- and z-axes, respectively. There is no x-component because the x-axis is perpendicular to the direction of gravity.

$$W_y = mg \sin\theta$$

$$W_z = mg \cos\theta$$

According to Newton's second law, the sum of the forces acting on the puck is equal to its mass times acceleration.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_x = ma_x \\ \sum F_y = ma_y \\ \sum F_z = ma_z \end{cases}$$

The puck stays in the xy-plane, so the sum of the forces in the z-direction is zero. Also, $\mathbf{W}_y$ points in the negative y-direction, so there's a minus sign on the left side of the y-equation.

$$\begin{cases} 0 = ma_x \\ -mg\sin\theta = ma_y \\ 0 = ma_z \end{cases}$$

Divide both sides of each equation by m.

$$\begin{cases} 0 = a_x \\ -g\sin\theta = a_y \\ 0 = a_z \end{cases}$$

Acceleration is the second derivative of position.

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\ \dfrac{d^2y}{dt^2} = -g\sin\theta \\ \dfrac{d^2z}{dt^2} = 0 \end{cases}$$

Integrate both sides of each equation with respect to t to get the components of the puck's velocity.

$$\begin{cases} \dfrac{dx}{dt} = C_1 \\ \dfrac{dy}{dt} = -gt\sin\theta + C_2 \\ \dfrac{dz}{dt} = C_3 \end{cases} \tag{1}$$

Use the puck's initial velocity vector $\mathbf{v}_\text{o} = \langle v_{\text{o}x}, v_{\text{o}y}, 0 \rangle$ to determine C_1, C_2, and C_3.

$$\begin{aligned}
\frac{dx}{dt}(0) &= C_1 = v_{\text{o}x} & &\rightarrow & C_1 &= v_{\text{o}x} \\
\frac{dy}{dt}(0) &= -g(0)\sin\theta + C_2 = v_{\text{o}y} & &\rightarrow & C_2 &= v_{\text{o}y} \\
\frac{dz}{dt}(0) &= C_3 = 0 & &\rightarrow & C_3 &= 0
\end{aligned}$$

As a result, equation (1) becomes

$$\begin{cases}
\dfrac{dx}{dt} = v_{\text{o}x} \\
\dfrac{dy}{dt} = -gt\sin\theta + v_{\text{o}y} \\
\dfrac{dz}{dt} = 0
\end{cases}.$$

Integrate both sides of each equation with respect to t once more to get the components of the puck's position.

$$\begin{cases}
x(t) = v_{\text{o}x}t + C_4 \\
y(t) = -\dfrac{gt^2}{2}\sin\theta + v_{\text{o}y}t + C_5 \\
z(t) = C_6
\end{cases} \tag{2}$$

Use the fact that the puck starts from the origin ($x = 0$, $y = 0$, and $z = 0$ when $t = 0$) to determine C_4, C_5, and C_6.

$$\begin{aligned}
x(0) &= v_{\text{o}x}(0) + C_4 = 0 & &\rightarrow & C_4 &= 0 \\
y(0) &= -\frac{g(0)^2}{2}\sin\theta + v_{\text{o}y}(0) + C_5 = 0 & &\rightarrow & C_5 &= 0 \\
z(0) &= C_6 = 0 & &\rightarrow & C_6 &= 0
\end{aligned}$$

Consequently, equation (2) becomes

$$\begin{cases}
x(t) = v_{\text{o}x}t \\
y(t) = -\dfrac{gt^2}{2}\sin\theta + v_{\text{o}y}t \\
z(t) = 0
\end{cases}.$$

Therefore, the puck's position is

$$\mathbf{r}(t) = \left\langle v_{\text{o}x}t, -\frac{gt^2}{2}\sin\theta + v_{\text{o}y}t, 0 \right\rangle.$$

To find how long it takes for the puck to return to the floor level, set $y(t) = 0$ and solve for nonzero t.

$$-\frac{gt^2}{2}\sin\theta + v_{\text{o}y}t = 0$$

$$t\left(-\frac{gt}{2}\sin\theta + v_{\text{o}y}\right) = 0$$

$$t = 0 \quad \text{or} \quad -\frac{gt}{2}\sin\theta + v_{\text{o}y} = 0$$

$$t = 0 \quad \text{or} \quad \boxed{t = \frac{2v_{\text{o}y}}{g\sin\theta}}$$

To find how far the puck is from O when it returns to the floor, plug this time into $x(t)$.

$$x\left(\frac{2v_{\text{o}y}}{g\sin\theta}\right) = v_{\text{o}x}\left(\frac{2v_{\text{o}y}}{g\sin\theta}\right) = \frac{2v_{\text{o}x}v_{\text{o}y}}{g\sin\theta}$$