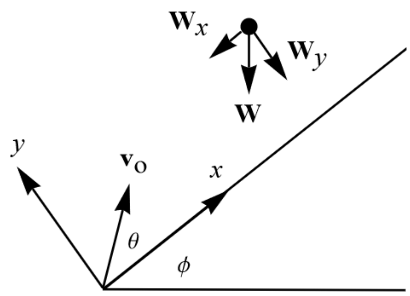
Problem 1.39

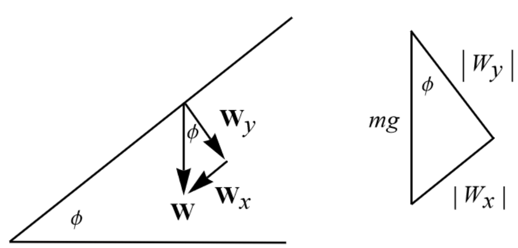
A ball is thrown with initial speed v_o up an inclined plane. The plane is inclined at an angle ϕ above the horizontal, and the ball's initial velocity is at an angle θ above the plane. Choose axes with x measured up the slope, y normal to the slope, and z across it. Write down Newton's second law using these axes and find the ball's position as a function of time. Show that the ball lands a distance $R = 2v_o^2 \sin\theta\cos(\theta+\phi)/(g\cos^2\phi)$ from its launch point. Show that for given v_o and ϕ, the maximum possible range up the inclined plane is $R_{\max} = v_o^2/[g(1+\sin\phi)]$.

Solution

Start by drawing the free-body diagram for the ball, noting that only the force of gravity acts on it. The weight vector **W** and its components along the x- and y-axes, $\mathbf{W}_x$ and $\mathbf{W}_y$, are shown.



Draw the right triangle formed by the weight vector and its components.



Use trigonometry to determine $|W_x|$ and $|W_y|$.

$$|W_x| = mg\sin\phi$$

$$|W_y| = mg\cos\phi$$

Since the weight vector components point in the negative x- and y-directions, both the components have minus signs.

$$W_x = -mg\sin\phi$$

$$W_y = -mg\cos\phi$$

According to Newton's second law, the sum of the forces acting on the ball is equal to its mass times acceleration.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_x = ma_x \\ \sum F_y = ma_y \\ \sum F_z = ma_z \end{cases}$$

The ball stays in the xy-plane, so the sum of the forces in the z-direction is zero.

$$\begin{cases} -mg\sin\phi = ma_x \\ -mg\cos\phi = ma_y \\ 0 = ma_z \end{cases}$$

Divide both sides of each equation by m.

$$\begin{cases} -g\sin\phi = a_x \\ -g\cos\phi = a_y \\ 0 = a_z \end{cases}$$

Acceleration is the second derivative of position.

$$\begin{cases} \dfrac{d^2x}{dt^2} = -g\sin\phi \\ \dfrac{d^2y}{dt^2} = -g\cos\phi \\ \dfrac{d^2z}{dt^2} = 0 \end{cases}$$

Integrate both sides of each equation with respect to t to get the components of the ball's velocity.

$$\begin{cases} \dfrac{dx}{dt} = -gt\sin\phi + C_1 \\ \dfrac{dy}{dt} = -gt\cos\phi + C_2 \\ \dfrac{dz}{dt} = C_3 \end{cases} \tag{1}$$

Use the ball's initial velocity vector $\mathbf{v}_o = \langle v_o\cos\theta, v_o\sin\theta, 0\rangle$ to determine C_1, C_2, and C_3.

$$\frac{dx}{dt}(0) = -g(0)\sin\phi + C_1 = v_o\cos\theta \quad \rightarrow \quad C_1 = v_o\cos\theta$$

$$\frac{dy}{dt}(0) = -g(0)\cos\phi + C_2 = v_o\sin\theta \quad \rightarrow \quad C_2 = v_o\sin\theta$$

$$\frac{dz}{dt}(0) = C_3 = 0 \quad \rightarrow \quad C_3 = 0$$

As a result, equation (1) becomes

$$\begin{cases} \dfrac{dx}{dt} = -gt\sin\phi + v_\text{o}\cos\theta \\ \dfrac{dy}{dt} = -gt\cos\phi + v_\text{o}\sin\theta \\ \dfrac{dz}{dt} = 0 \end{cases}.$$

Integrate both sides of each equation with respect to t once more to get the components of the ball's position.

$$\begin{cases} x(t) = -\dfrac{1}{2}gt^2\sin\phi + v_\text{o}t\cos\theta + C_4 \\ y(t) = -\dfrac{1}{2}gt^2\cos\phi + v_\text{o}t\sin\theta + C_5 \\ z(t) = C_6 \end{cases} \tag{2}$$

Use the fact that the ball starts from the origin ($x = 0$, $y = 0$, and $z = 0$ when $t = 0$) to determine C_4, C_5, and C_6.

$$x(0) = -\frac{1}{2}g(0)^2\sin\phi + v_\text{o}(0)\cos\theta + C_4 = 0 \quad \rightarrow \quad C_4 = 0$$

$$y(0) = -\frac{1}{2}g(0)^2\cos\phi + v_\text{o}(0)\sin\theta + C_5 = 0 \quad \rightarrow \quad C_5 = 0$$

$$z(0) = C_6 = 0 \quad \rightarrow \quad C_6 = 0$$

Consequently, equation (2) becomes

$$\begin{cases} x(t) = -\dfrac{1}{2}gt^2\sin\phi + v_\text{o}t\cos\theta \\ y(t) = -\dfrac{1}{2}gt^2\cos\phi + v_\text{o}t\sin\theta \\ z(t) = 0 \end{cases}.$$

Therefore, the ball's position is

$$\boxed{\mathbf{r}(t) = \left\langle -\frac{1}{2}gt^2\sin\phi + v_\text{o}t\cos\theta, -\frac{1}{2}gt^2\cos\phi + v_\text{o}t\sin\theta, 0 \right\rangle}.$$

To find how long it takes for the ball to return to the inclined plane, set $y(t) = 0$ and solve for nonzero t.

$$y(t) = 0$$

$$-\frac{1}{2}gt^2\cos\phi + v_o t\sin\theta = 0$$

$$t\left(-\frac{1}{2}gt\cos\phi + v_o\sin\theta\right) = 0$$

$$t = 0 \quad \text{or} \quad -\frac{1}{2}gt\cos\phi + v_o\sin\theta = 0$$

$$t = 0 \quad \text{or} \quad t = \frac{2v_o\sin\theta}{g\cos\phi}$$

To find how far the ball is from the origin when it returns to the inclined plane, plug this time into $x(t)$.

$$\begin{aligned} x\left(\frac{2v_o\sin\theta}{g\cos\phi}\right) &= -\frac{1}{2}g\left(\frac{2v_o\sin\theta}{g\cos\phi}\right)^2\sin\phi + v_o\left(\frac{2v_o\sin\theta}{g\cos\phi}\right)\cos\theta \\ &= -\frac{1}{2}g\left(\frac{4v_o^2\sin^2\theta}{g^2\cos^2\phi}\right)\sin\phi + v_o\left(\frac{2v_o\sin\theta}{g\cos\phi}\right)\cos\theta \\ &= -\frac{2v_o^2\sin^2\theta\sin\phi}{g\cos^2\phi} + \frac{2v_o^2\sin\theta\cos\theta}{g\cos\phi} \\ &= \frac{-2v_o^2\sin^2\theta\sin\phi + 2v_o^2\sin\theta\cos\theta\cos\phi}{g\cos^2\phi} \\ &= \frac{2v_o^2\sin\theta(-\sin\theta\sin\phi + \cos\theta\cos\phi)}{g\cos^2\phi} \\ &= \frac{2v_o^2\sin\theta\cos(\theta+\phi)}{g\cos^2\phi} \end{aligned}$$

This is how far the ball lands from the launch point, the range.

$$\boxed{R = \frac{2v_o^2\sin\theta\cos(\theta+\phi)}{g\cos^2\phi}}$$

To determine the largest range for given v_o and ϕ, take the derivative with respect to θ,

$$R'(\theta) = \frac{d}{d\theta}\left[\frac{2v_o^2\sin\theta\cos(\theta+\phi)}{g\cos^2\phi}\right] = \frac{2v_o^2\cos\theta\cos(\theta+\phi)}{g\cos^2\phi} + \frac{2v_o^2\sin\theta[-\sin(\theta+\phi)]}{g\cos^2\phi}$$

set the result equal to zero, and solve the equation for θ.

$$\frac{2v_o^2\cos\theta\cos(\theta+\phi)}{g\cos^2\phi}+\frac{2v_o^2\sin\theta[-\sin(\theta+\phi)]}{g\cos^2\phi}=0$$

$$\frac{2v_o^2}{g\cos^2\phi}\left[\cos\theta\cos(\theta+\phi)-\sin\theta\sin(\theta+\phi)\right]=0$$

$$\frac{2v_o^2}{g\cos^2\phi}\cos\left[\theta+(\theta+\phi)\right]=0$$

$$\frac{2v_o^2}{g\cos^2\phi}\cos(2\theta+\phi)=0$$

$$\cos(2\theta+\phi)=0$$

$$2\theta+\phi=\frac{1}{2}(2n-1)\pi,\quad n=0,\pm1,\pm2,\ldots$$

The appropriate value to choose on the right side is the first one greater than zero.

$$2\theta+\phi=\frac{\pi}{2}$$

$$2\theta=\frac{\pi}{2}-\phi$$

$$\theta_{\text{max}}=\frac{\pi}{4}-\frac{\phi}{2}$$

Plug this value of θ into the formula for R to get the maximum range.

$$R(\theta_{\text{max}})=\frac{2v_o^2\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cos\left[\left(\frac{\pi}{4}-\frac{\phi}{2}\right)+\phi\right]}{g\cos^2\phi}$$

$$R_{\text{max}}=\frac{2v_o^2\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)}{g\cos^2\phi}$$

$$=\frac{2v_o^2\left(\sin\frac{\pi}{4}\cos\frac{\phi}{2}-\cos\frac{\pi}{4}\sin\frac{\phi}{2}\right)\left(\cos\frac{\pi}{4}\cos\frac{\phi}{2}-\sin\frac{\pi}{4}\sin\frac{\phi}{2}\right)}{g\cos^2\phi}$$

$$=\frac{2v_o^2\left(\frac{1}{\sqrt{2}}\cos\frac{\phi}{2}-\frac{1}{\sqrt{2}}\sin\frac{\phi}{2}\right)\left(\frac{1}{\sqrt{2}}\cos\frac{\phi}{2}-\frac{1}{\sqrt{2}}\sin\frac{\phi}{2}\right)}{g\cos^2\phi}$$

$$=\frac{2v_o^2\left(\frac{1}{2}\cos^2\frac{\phi}{2}-\sin\frac{\phi}{2}\cos\frac{\phi}{2}+\frac{1}{2}\sin^2\frac{\phi}{2}\right)}{g\cos^2\phi}$$

$$=\frac{v_o^2\left(\cos^2\frac{\phi}{2}-2\sin\frac{\phi}{2}\cos\frac{\phi}{2}+\sin^2\frac{\phi}{2}\right)}{g\cos^2\phi}$$

$$=\frac{v_o^2\left(1-2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)}{g\cos^2\phi}$$

Continue the simplification.

$$\begin{aligned}
R_{\text{max}} &= \frac{v_o^2(1-\sin\phi)}{g\cos^2\phi} \\
&= \frac{v_o^2(1-\sin\phi)}{g(1-\sin^2\phi)} \\
&= \frac{v_o^2(1-\sin\phi)}{g(1+\sin\phi)(1-\sin\phi)} \\
&= \frac{v_o^2}{g(1+\sin\phi)}
\end{aligned}$$

Therefore, the maximum possible range up the inclined plane is

$$\boxed{R_{\text{max}} = \frac{v_o^2}{g(1+\sin\phi)},}$$

and it occurs if $\theta = \pi/4 - \phi/2$.