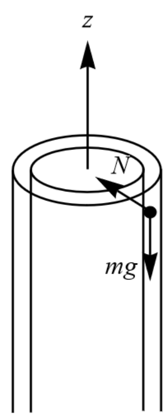
Problem 1.49

Imagine two concentric cylinders, centered on the vertical z axis, with radii $R \pm \epsilon$, where ϵ is very small. A small frictionless puck of thickness 2ϵ is inserted between the two cylinders, so that it can be considered a point mass that can move freely at a fixed distance from the vertical axis. If we use cylindrical polar coordinates (ρ, ϕ, z) for its position (Problem 1.47), then ρ is fixed at $\rho = R$, while ϕ and z can vary at will. Write down and solve Newton's second law for the general motion of the puck, including the effects of gravity. Describe the puck's motion.

Solution

Start by drawing a free-body diagram for the puck, which moves freely between the lateral sides of two coaxial cylinders.



The gravitational force and a normal force act on the puck. Without the normal force, provided by the cylinder with radius $R + \epsilon$, the puck would fly out of bounds. Newton's second law states that the sum of the forces on the mass is equal to its mass times acceleration.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_r = ma_r \\ \sum F_\phi = ma_\phi \\ \sum F_z = ma_z \end{cases}$$

The gravitational force acts in the negative z-direction, and the normal force acts in the negative r-direction.

$$\begin{cases} -N = ma_r \\ 0 = ma_\phi \\ -mg = ma_z \end{cases}$$

Divide both sides of each equation by m.

$$\begin{cases} -\frac{N}{m} = a_r \\ 0 = a_\phi \\ -g = a_z \end{cases}$$

Substitute the formulas for acceleration in cylindrical coordinates.

$$\begin{cases} \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\phi}{dt}\right)^2 = -\dfrac{N}{m} \\ r\dfrac{d^2\phi}{dt^2} + 2\dfrac{dr}{dt}\dfrac{d\phi}{dt} = 0 \\ \dfrac{d^2z}{dt^2} = -g \end{cases}$$

Because the puck is constrained to move between the cylinders, $r = R$, $dr/dt = 0$, and $d^2r/dt^2 = 0$.

$$\begin{cases} 0 - R\left(\dfrac{d\phi}{dt}\right)^2 = -\dfrac{N}{m} \\ R\dfrac{d^2\phi}{dt^2} + 2(0)\dfrac{d\phi}{dt} = 0 \\ \dfrac{d^2z}{dt^2} = -g \end{cases}$$

$$\begin{cases} \left(\dfrac{d\phi}{dt}\right)^2 = \dfrac{N}{mR} \\ \dfrac{d^2\phi}{dt^2} = 0 \\ \dfrac{d^2z}{dt^2} = -g \end{cases}$$

Integrate both sides of the last two equations with respect to t.

$$\begin{cases} \left(\dfrac{d\phi}{dt}\right)^2 = \dfrac{N}{mR} \\ \dfrac{d\phi}{dt} = C_1 \\ \dfrac{dz}{dt} = -gt + C_2 \end{cases} \tag{1}$$

In order to determine C_1 and C_2, assume that, at $t = 0$, the puck is moving with angular velocity ω about the z-axis and linear velocity v_{0z} in the z-direction.

$$\frac{d\phi}{dt}(0) = C_1 = \omega \quad \rightarrow \quad C_1 = \omega$$

$$\frac{dz}{dt}(0) = -g(0) + C_2 = v_{0z} \quad \rightarrow \quad C_2 = v_{0z}$$

As a result, equation (1) becomes

$$\begin{cases} \left(\frac{d\phi}{dt}\right)^2 = \frac{N}{mR} \\ \frac{d\phi}{dt} = \omega \\ \frac{dz}{dt} = -gt + v_{0z} \end{cases}.$$

Integrate the last two equations with respect to t once more.

$$\begin{cases} \left(\frac{d\phi}{dt}\right)^2 = \frac{N}{mR} \\ \phi(t) = \omega t + C_3 \\ z(t) = -\frac{1}{2}gt^2 + v_{0z}t + C_4 \end{cases} \tag{2}$$

In order to determine C_3 and C_4, assume additionally that, at $t = 0$, the puck is located at $\phi = \phi_0$ and $z = z_0$.

$$\phi(0) = \omega(0) + C_3 = \phi_0 \quad \rightarrow \quad C_3 = \phi_0$$

$$z(0) = -\frac{1}{2}g(0)^2 + v_{0z}(0) + C_4 = z_0 \quad \rightarrow \quad C_4 = z_0$$

As a result, equation (2) becomes

$$\begin{cases} \left(\frac{d\phi}{dt}\right)^2 = \frac{N}{mR} \\ \phi(t) = \omega t + \phi_0 \\ z(t) = -\frac{1}{2}gt^2 + v_{0z}t + z_0 \end{cases}.$$

If we wanted to know the normal force, we could use the first equation.

$$(\omega)^2 = \frac{N}{mR} \quad \rightarrow \quad N = mR\omega^2$$

Therefore, the equations describing the puck's motion are

$$\begin{cases} r(t) = R \\ \phi(t) = \omega t + \phi_0 \\ z(t) = -\frac{1}{2}gt^2 + v_{0z}t + z_0 \end{cases}.$$

The puck moves in the shape of a helix that keeps elongating in the z-direction.