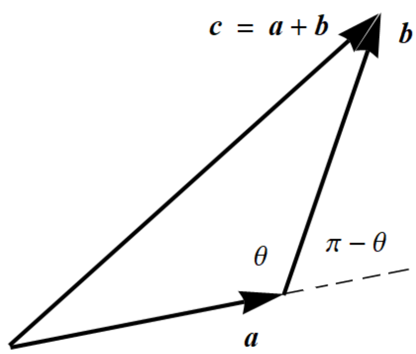
Problem 1.9

In elementary trigonometry, you probably learned the law of cosines for a triangle of sides a, b, and c, that $c^2 = a^2 + b^2 - 2ab\cos\theta$, where θ is the angle between the sides a and b. Show that the law of cosines is an immediate consequence of the identity $(\mathbf{a}+\mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b}$.

Solution

$$\begin{aligned}(\mathbf{a}+\mathbf{b})^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\\ &= \mathbf{a}\cdot(\mathbf{a}+\mathbf{b}) + \mathbf{b}\cdot(\mathbf{a}+\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}\\ &= a^2 + \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b} + b^2\\ &= a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b}\end{aligned}$$

Consider the triangle formed by the two vectors, $\mathbf{a}$ and $\mathbf{b}$, and their sum. Let θ be the angle opposite to the sum. Then the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi - \theta$.



Substitute $\mathbf{c}$ for $\mathbf{a}+\mathbf{b}$ and use the definition of the dot product on the right side.

$$\begin{aligned}(\mathbf{c})^2 &= a^2 + b^2 + 2ab\cos(\pi-\theta)\\ \mathbf{c}\cdot\mathbf{c} &= a^2 + b^2 + 2ab(\cos\pi\cos\theta + \sin\pi\sin\theta)\\ c^2 &= a^2 + b^2 + 2ab(-\cos\theta)\end{aligned}$$

Therefore, the law of cosines,

$$c^2 = a^2 + b^2 - 2ab\cos\theta,$$

is a consequence.