Problem 16.10

Using the integral (16.33), show that the Fourier coefficients of the triangular wave of Figure 16.7 are zero for n even and given by (16.34) for n odd.

Solution

The triangular wave is shown in Figure 16.7.

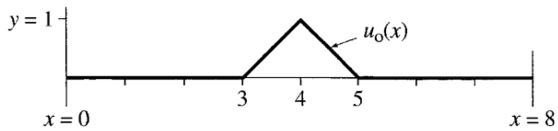

Figure 16.7 A string is released from rest at $t = 0$ in the triangular position shown.

Write down the formula for $u_o(x)$. Notice that it's the graph of $y = 1 - |x|$ but translated to the right by 4 units.

$$u_o(x) = \begin{cases} 1 - |x-4| & \text{if } |x-4| \leq 1 \\ 0 & \text{if } |x-4| > 1 \end{cases}$$

This function is nonzero if

$$|x-4| \leq 1$$

$$-1 \leq x - 4 \leq 1$$

$$3 \leq x \leq 5$$

as expected. The formula for the Fourier coefficients is given in (16.33) on page 692.

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L u_o(x)\sin\frac{n\pi x}{L}\,dx \qquad (16.33) \\ &= \frac{2}{8}\int_0^8 u_o(x)\sin\frac{n\pi x}{8}\,dx \\ &= \frac{1}{4}\int_3^5 (1-|x-4|)\sin\frac{n\pi x}{8}\,dx \end{aligned}$$

In order to make the integration interval symmetric, make the following substitution.

$$u = x - 4 \quad \to \quad x = u + 4$$

$$du = dx$$

As a result,

$$\begin{aligned} B_n &= \frac{1}{4}\int_{3-4}^{5-4} (1-|u|)\sin\frac{n\pi(u+4)}{8}\,du \\ &= \frac{1}{4}\int_{-1}^{1} (1-|u|)\sin\left(\frac{n\pi u}{8} + \frac{n\pi}{2}\right)du. \end{aligned}$$

Use the angle addition formula for sine and then split up the integral.

$$\begin{aligned}
B_n &= \frac{1}{4}\int_{-1}^{1}(1-|u|)\left(\sin\frac{n\pi u}{8}\cos\frac{n\pi}{2}+\cos\frac{n\pi u}{8}\sin\frac{n\pi}{2}\right)du \\
&= \frac{1}{4}\Bigg[\underbrace{\int_{-1}^{1}(1-|u|)\sin\frac{n\pi u}{8}\cos\frac{n\pi}{2}\,du}_{=0} + \int_{-1}^{1}(1-|u|)\cos\frac{n\pi u}{8}\sin\frac{n\pi}{2}\,du\Bigg]
\end{aligned}$$

This first integral is zero because the integration interval is symmetric and the integrand is an odd function. The integrand of the second integral is an even function, so the integration interval can be made to go from 0 to 1 as long as a factor of 2 is placed in front.

$$\begin{aligned}
B_n &= \frac{1}{4}\left[2\int_0^1(1-|u|)\cos\frac{n\pi u}{8}\sin\frac{n\pi}{2}\,du\right] \\
&= \frac{1}{2}\sin\frac{n\pi}{2}\int_0^1(1-u)\cos\frac{n\pi u}{8}\,du \\
&= \frac{1}{2}\sin\frac{n\pi}{2}\left(\int_0^1\cos\frac{n\pi u}{8}\,du-\int_0^1 u\cos\frac{n\pi u}{8}\,du\right) \\
&= \frac{1}{2}\sin\frac{n\pi}{2}\left[\frac{8}{n\pi}\sin\frac{n\pi u}{8}\bigg|_0^1-\int_0^1\frac{\partial}{\partial k}(\sin ku)\bigg|_{k=n\pi/8}du\right] \\
&= \frac{1}{2}\sin\frac{n\pi}{2}\left[\frac{8}{n\pi}\sin\frac{n\pi}{8}-\frac{d}{dk}\left(\int_0^1\sin ku\,du\right)\bigg|_{k=n\pi/8}\right] \\
&= \frac{1}{2}\sin\frac{n\pi}{2}\left[\frac{8}{n\pi}\sin\frac{n\pi}{8}-\frac{d}{dk}\left(-\frac{1}{k}\cos ku\bigg|_0^1\right)\bigg|_{k=n\pi/8}\right] \\
&= \frac{1}{2}\sin\frac{n\pi}{2}\left[\frac{8}{n\pi}\sin\frac{n\pi}{8}+\frac{d}{dk}\left(\frac{\cos k-1}{k}\right)\bigg|_{k=n\pi/8}\right] \\
&= \frac{1}{2}\sin\frac{n\pi}{2}\left[\frac{8}{n\pi}\sin\frac{n\pi}{8}+\left(\frac{1-\cos k-k\sin k}{k^2}\right)\bigg|_{k=n\pi/8}\right] \\
&= \frac{1}{2}\sin\frac{n\pi}{2}\left[\frac{8}{n\pi}\sin\frac{n\pi}{8}+\frac{64}{n^2\pi^2}\left(1-\cos\frac{n\pi}{8}-\frac{n\pi}{8}\sin\frac{n\pi}{8}\right)\right] \\
&= \frac{1}{2}\sin\frac{n\pi}{2}\left[\cancel{\frac{8}{n\pi}\sin\frac{n\pi}{8}}+\frac{64}{n^2\pi^2}\left(1-\cos\frac{n\pi}{8}\right)-\cancel{\frac{8}{n\pi}\sin\frac{n\pi}{8}}\right] \\
&= \frac{32}{n^2\pi^2}\sin\frac{n\pi}{2}\left(1-\cos\frac{n\pi}{8}\right)
\end{aligned}$$

Because of the $\sin(n\pi/2)$ factor, B_n is zero if n is even. To simplify the formula, then, set $n = 2m+1$, where $m = 0, 1, 2, \ldots$.

$$B_{2m+1} = \frac{32}{(2m+1)^2\pi^2}\overbrace{\sin\frac{(2m+1)\pi}{2}}^{=(-1)^m}\left[1-\cos\frac{(2m+1)\pi}{8}\right]$$

Therefore,

$$B_{2m+1} = (-1)^m \frac{32}{(2m+1)^2\pi^2}\left[1 - \cos\frac{(2m+1)\pi}{8}\right].$$