Problem 16.13

In connection with Equation (16.31), I claimed that *any* function on the interval $0 \le x \le L$ can be expanded in a Fourier series containing just sine functions. This is at first sight very surprising since one is used to the claim that the general Fourier series requires sines *and* cosines. In this problem, you'll prove this surprising claim. Let $f(x)$ be any function defined for $0 \le x \le L$. We can define a function $f(x)$ for *all* x by setting it equal to the given function in the original interval and requiring that

$$f(-x) = -f(x) \qquad \text{and} \qquad f(x+2L) = f(x). \tag{16.143}$$

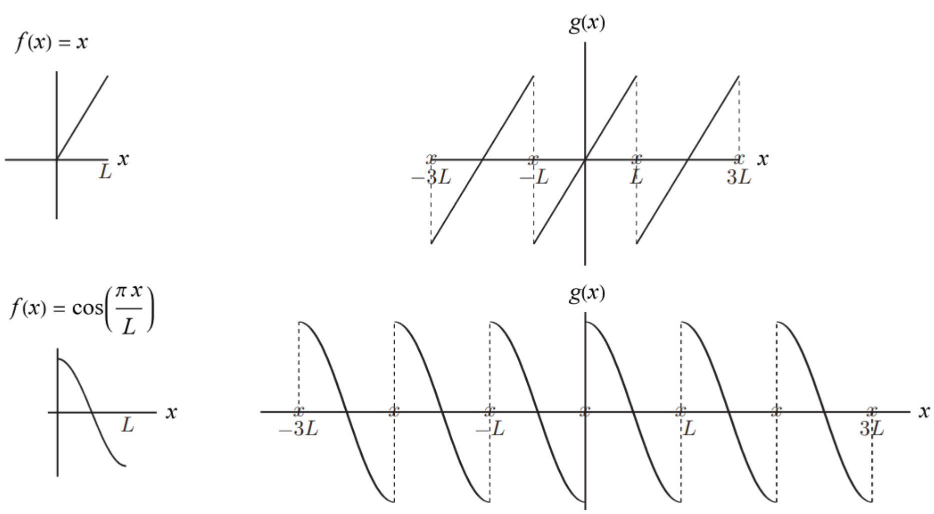
for all x. Prove that this defines a function which is (1) periodic with period $2L$, (2) odd, and (3) the same as the original $f(x)$ on the original interval. Write down the ordinary Fourier expansion for this new $f(x)$ and show that the coefficients of the cosine terms are all zero. This establishes the possibility of expanding the original function in terms of sines alone.[27] Bearing in mind that the period of the new function is $2L$, write down the standard formula (5.84) for the expansion coefficients and show that your answer agrees with (16.33). The Fourier sine series is especially convenient for discussing functions that are zero at the end points $x = 0$ and L.

Solution

Let $f(x)$ be a function defined over $0 \le x \le L$. Then define another function $g(x)$ over the whole line ($-\infty < x < \infty$) as follows.

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L \\ g(-x) = -g(x) & \\ g(x+2L) = g(x) & \end{cases}$$

Below are some examples of $f(x)$ and the resulting graphs of $g(x)$.



[27] But note that it has the form $\sum B_n \sin(n\pi x/L)$. The usual Fourier series has sines and cosines, but their argument is $2n\pi x/L$. Thus the new Fourier sine series has, in a sense, twice as many terms to make up for having only sines.

The condition $f(x)$ if $0 \leq x \leq L$ means that the function looks like $f(x)$ on the original interval, the condition $g(-x) = -g(x)$ means that $g(x)$ is an odd function, and the condition $g(x + 2L) = g(x)$ means that $g(x)$ is $2L$-periodic. Because $g(x)$ is defined on the whole line and is $2L$-periodic, it has a Fourier series expansion.

$$g(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \tag{1}$$

To determine A_0, integrate both sides with respect to x from $-L$ to L.

$$\begin{aligned}
\int_{-L}^{L} g(x)\,dx &= \int_{-L}^{L} \left(A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \right) dx \\
&= \int_{-L}^{L} A_0\,dx + \int_{-L}^{L} \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\,dx + \int_{-L}^{L} \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\,dx \\
&= A_0 \int_{-L}^{L} dx + \sum_{n=1}^{\infty} A_n \int_{-L}^{L} \cos\frac{n\pi x}{L}\,dx + \sum_{n=1}^{\infty} B_n \int_{-L}^{L} \sin\frac{n\pi x}{L}\,dx \\
&= A_0(2L) + \sum_{n=1}^{\infty} A_n(0) + \sum_{n=1}^{\infty} B_n(0) \\
&= 2LA_0
\end{aligned}$$

Divide both sides by $2L$ and then evaluate the remaining integral.

$$\begin{aligned}
A_0 &= \frac{1}{2L} \int_{-L}^{L} g(x)\,dx \\
&= \frac{1}{2L} \left[\int_{-L}^{0} g(x)\,dx + \int_{0}^{L} g(x)\,dx \right] \\
&= \frac{1}{2L} \left[\int_{L}^{0} g(-x)\,(-dx) + \int_{0}^{L} g(x)\,dx \right] \\
&= \frac{1}{2L} \left[\int_{0}^{L} g(-x)\,dx + \int_{0}^{L} g(x)\,dx \right] \\
&= \frac{1}{2L} \left[\int_{0}^{L} [-g(x)]\,dx + \int_{0}^{L} g(x)\,dx \right] \\
&= \frac{1}{2L} \left[-\int_{0}^{L} g(x)\,dx + \int_{0}^{L} g(x)\,dx \right] \\
&= \frac{1}{2L}(0) \\
&= 0
\end{aligned}$$

To determine A_n, multiply both sides of equation (1) by $\cos(m\pi x/L)$, where m is another positive integer,

$$g(x)\cos\frac{m\pi x}{L} = A_0\cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$$

and then integrate both sides with respect to x from $-L$ to L.

$$\begin{aligned}
\int_{-L}^{L} g(x)\cos\frac{m\pi x}{L}\,dx &= \int_{-L}^{L}\left(A_0\cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right)dx \\
&= \int_{-L}^{L} A_0\cos\frac{m\pi x}{L}\,dx + \int_{-L}^{L}\sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx + \int_{-L}^{L}\sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \\
&= A_0\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx + \sum_{n=1}^{\infty} A_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx + \sum_{n=1}^{\infty} B_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \\
&= A_0(0) + \sum_{n=1}^{\infty} A_n\int_{-L}^{L}\frac{1}{2}\left[\cos\left(\frac{n\pi x}{L} - \frac{m\pi x}{L}\right) + \cos\left(\frac{n\pi x}{L} + \frac{m\pi x}{L}\right)\right]dx \\
&\qquad + \sum_{n=1}^{\infty} B_n\int_{-L}^{L}\frac{1}{2}\left[\sin\left(\frac{n\pi x}{L} + \frac{m\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L} - \frac{m\pi x}{L}\right)\right]dx \\
&= \sum_{n=1}^{\infty}\frac{A_n}{2}\int_{-L}^{L}\left[\cos\frac{(n-m)\pi x}{L} + \cos\frac{(n+m)\pi x}{L}\right]dx \\
&\qquad + \sum_{n=1}^{\infty}\frac{B_n}{2}\int_{-L}^{L}\left[\sin\frac{(n+m)\pi x}{L} + \sin\frac{(n-m)\pi x}{L}\right]dx \qquad (2)
\end{aligned}$$

If $n \neq m$, then the integrals evaluate to

$$\begin{aligned}
\int_{-L}^{L}\left[\cos\frac{(n-m)\pi x}{L} + \cos\frac{(n+m)\pi x}{L}\right]dx &= \frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L}\bigg|_{-L}^{L} + \frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L}\bigg|_{-L}^{L} \\
&= \frac{L}{(n-m)\pi}(0-0) + \frac{L}{(n+m)\pi}(0-0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\int_{-L}^{L}\left[\sin\frac{(n+m)\pi x}{L} + \sin\frac{(n-m)\pi x}{L}\right] &= \frac{-L}{(n+m)\pi}\cos\frac{(n+m)\pi x}{L}\bigg|_{-L}^{L} + \frac{-L}{(n-m)\pi}\cos\frac{(n-m)\pi x}{L}\bigg|_{-L}^{L} \\
&= \frac{-L}{(n+m)\pi}[\cos(n+m)\pi - \cos(n+m)\pi] \\
&\qquad + \frac{-L}{(n-m)\pi}[\cos(n-m)\pi - \cos(n-m)\pi] \\
&= 0.
\end{aligned}$$

Consequently, every term in each infinite series in equation (2) vanishes except for one—the term in which $n = m$.

$$\begin{aligned}
\int_{-L}^{L} g(x)\cos\frac{n\pi x}{L}\,dx &= \frac{A_n}{2}\int_{-L}^{L}\left[\cos\frac{(n-n)\pi x}{L} + \cos\frac{(n+n)\pi x}{L}\right]dx \\
&\qquad + \frac{B_n}{2}\int_{-L}^{L}\left[\sin\frac{(n+n)\pi x}{L} + \sin\frac{(n-n)\pi x}{L}\right]dx \\
&= \frac{A_n}{2}\int_{-L}^{L}\left(1+\cos\frac{2n\pi x}{L}\right)dx \\
&\qquad + \frac{B_n}{2}\int_{-L}^{L}\left(\sin\frac{2n\pi x}{L} + 0\right)dx \\
&= \frac{A_n}{2}\int_{-L}^{L} dx + \frac{A_n}{2}\int_{-L}^{L}\cos\frac{2n\pi x}{L}\,dx + \frac{B_n}{2}\int_{-L}^{L}\sin\frac{2n\pi x}{L}\,dx \\
&= \frac{A_n}{2}(2L) + \frac{A_n}{2}(0) + \frac{B_n}{2}(0) \\
&= LA_n
\end{aligned}$$

Divide both sides by L and then evaluate the remaining integral.

$$\begin{aligned}
A_n &= \frac{1}{L}\int_{-L}^{L} g(x)\cos\frac{n\pi x}{L}\,dx \\
&= \frac{1}{L}\left[\int_{-L}^{0} g(x)\cos\frac{n\pi x}{L}\,dx + \int_{0}^{L} g(x)\cos\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}\left[\int_{L}^{0} g(-x)\cos\frac{n\pi(-x)}{L}\,(-dx) + \int_{0}^{L} g(x)\cos\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}\left[\int_{0}^{L} g(-x)\cos\left(-\frac{n\pi x}{L}\right)dx + \int_{0}^{L} g(x)\cos\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}\left[\int_{0}^{L} g(-x)\cos\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} g(x)\cos\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}\left[\int_{0}^{L} [-g(x)]\cos\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} g(x)\cos\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}\left[-\int_{0}^{L} g(x)\cos\frac{n\pi x}{L}\,dx + \int_{0}^{L} g(x)\cos\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}(0) \\
&= 0
\end{aligned}$$

To determine B_n, multiply both sides of equation (1) by $\sin(p\pi x/L)$, where p is another positive integer,

$$g(x)\sin\frac{p\pi x}{L} = A_0\sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}$$

and then integrate both sides with respect to x from $-L$ to L.

$$\begin{aligned}
\int_{-L}^{L} g(x)\sin\frac{p\pi x}{L}\,dx &= \int_{-L}^{L}\left(A_0\sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\right)dx \\
&= \int_{-L}^{L} A_0\sin\frac{p\pi x}{L}\,dx + \int_{-L}^{L}\sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx + \int_{-L}^{L}\sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx \\
&= A_0\int_{-L}^{L}\sin\frac{p\pi x}{L}\,dx + \sum_{n=1}^{\infty} A_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx + \sum_{n=1}^{\infty} B_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx \\
&= A_0(0) + \sum_{n=1}^{\infty} A_n\int_{-L}^{L}\frac{1}{2}\left[\sin\left(\frac{n\pi x}{L}+\frac{p\pi x}{L}\right) - \sin\left(\frac{n\pi x}{L}-\frac{p\pi x}{L}\right)\right]dx \\
&\qquad + \sum_{n=1}^{\infty} B_n\int_{-L}^{L}\frac{1}{2}\left[\cos\left(\frac{n\pi x}{L}-\frac{p\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L}+\frac{p\pi x}{L}\right)\right]dx \\
&= \sum_{n=1}^{\infty}\frac{A_n}{2}\int_{-L}^{L}\left[\sin\frac{(n+p)\pi x}{L} - \sin\frac{(n-p)\pi x}{L}\right]dx \\
&\qquad + \sum_{n=1}^{\infty}\frac{B_n}{2}\int_{-L}^{L}\left[\cos\frac{(n-p)\pi x}{L} - \cos\frac{(n+p)\pi x}{L}\right]dx \qquad (3)
\end{aligned}$$

If $n \neq p$, then the integrals evaluate to

$$\begin{aligned}
\int_{-L}^{L}\left[\cos\frac{(n-p)\pi x}{L} - \cos\frac{(n+p)\pi x}{L}\right]dx &= \frac{L}{(n-p)\pi}\sin\frac{(n-p)\pi x}{L}\bigg|_{-L}^{L} - \frac{L}{(n+p)\pi}\sin\frac{(n+p)\pi x}{L}\bigg|_{-L}^{L} \\
&= \frac{L}{(n-p)\pi}(0-0) - \frac{L}{(n+p)\pi}(0-0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\int_{-L}^{L}\left[\sin\frac{(n+p)\pi x}{L} - \sin\frac{(n-p)\pi x}{L}\right] &= \frac{-L}{(n+p)\pi}\cos\frac{(n+p)\pi x}{L}\bigg|_{-L}^{L} - \frac{-L}{(n-p)\pi}\cos\frac{(n-p)\pi x}{L}\bigg|_{-L}^{L} \\
&= \frac{-L}{(n+p)\pi}[\cos(n+p)\pi - \cos(n+p)\pi] \\
&\qquad - \frac{-L}{(n-p)\pi}[\cos(n-p)\pi - \cos(n-p)\pi] \\
&= 0.
\end{aligned}$$

Consequently, every term in each infinite series in equation (3) vanishes except for one—the term in which $n = p$.

$$\begin{aligned}
\int_{-L}^{L} g(x)\sin\frac{n\pi x}{L}\,dx &= \frac{A_n}{2}\int_{-L}^{L}\left[\sin\frac{(n+n)\pi x}{L} - \sin\frac{(n-n)\pi x}{L}\right]dx \\
&\qquad + \frac{B_n}{2}\int_{-L}^{L}\left[\cos\frac{(n-n)\pi x}{L} - \cos\frac{(n+n)\pi x}{L}\right]dx \\
&= \frac{A_n}{2}\int_{-L}^{L}\left(\sin\frac{2n\pi x}{L} - 0\right)dx \\
&\qquad + \frac{B_n}{2}\int_{-L}^{L}\left(1 - \cos\frac{2n\pi x}{L}\right)dx \\
&= \frac{A_n}{2}\int_{-L}^{L}\sin\frac{2n\pi x}{L}\,dx + \frac{B_n}{2}\int_{-L}^{L}dx - \frac{B_n}{2}\int_{-L}^{L}\cos\frac{2n\pi x}{L}\,dx \\
&= \frac{A_n}{2}(0) + \frac{B_n}{2}(2L) - \frac{B_n}{2}(0) \\
&= LB_n
\end{aligned}$$

Divide both sides by L and then evaluate the remaining integral.

$$\begin{aligned}
B_n &= \frac{1}{L}\int_{-L}^{L} g(x)\sin\frac{n\pi x}{L}\,dx \\
&= \frac{1}{L}\left[\int_{-L}^{0} g(x)\sin\frac{n\pi x}{L}\,dx + \int_{0}^{L} g(x)\sin\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}\left[\int_{L}^{0} g(-x)\sin\frac{n\pi(-x)}{L}\,(-dx) + \int_{0}^{L} g(x)\sin\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}\left[\int_{0}^{L} g(-x)\sin\left(-\frac{n\pi x}{L}\right)dx + \int_{0}^{L} g(x)\sin\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}\left[\int_{0}^{L} g(-x)\left(-\sin\frac{n\pi x}{L}\right)dx + \int_{0}^{L} g(x)\sin\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}\left[\int_{0}^{L} [-g(x)]\left(-\sin\frac{n\pi x}{L}\right)dx + \int_{0}^{L} g(x)\sin\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}\left[\int_{0}^{L} g(x)\sin\frac{n\pi x}{L}\,dx + \int_{0}^{L} g(x)\sin\frac{n\pi x}{L}\,dx\right] \\
&= \frac{1}{L}\left[2\int_{0}^{L} g(x)\sin\frac{n\pi x}{L}\,dx\right] \\
&= \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx
\end{aligned}$$

This result agrees with equation (16.33) on page 692.