Problem 16.14

[Computer] A taut string of length $L = 1$ is released from rest at $t = 0$, with initial position

$$u(x,0) = \begin{cases} 2x & [0 \le x \le \frac{1}{2}] \\ 2(1-x) & [\frac{1}{2} \le x \le 1]. \end{cases} \tag{16.144}$$

Take the wave speed on the string to be $c = 1$. **(a)** Sketch this initial shape and find the coefficients B_n in its Fourier sine series (16.31). **(b)** Make plots of the sum of the first several terms for several closely spaced times between $t = 0$ and τ, the period. Animate your plots and describe the motion.

Solution

In Problem 16.9 the general solution to the initial boundary value problem,

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ -\infty < t < \infty \\ &u(0,t) = 0 \\ &u(L,t) = 0 \\ &u(x,0) = f(x) \\ &\frac{\partial u}{\partial t}(x,0) = g(x), \end{aligned}$$

was found to be

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L},$$

where

$$\begin{aligned} A_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \\ B_n &= \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx. \end{aligned}$$

For a string released from rest initially with the prescribed shape the initial data are

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \dfrac{1}{2} \\ 2(1-x) & \text{if } \dfrac{1}{2} \le x \le 1 \end{cases}$$

$$g(x) = 0,$$

so the coefficients B_n evaluate to

$$B_n = \frac{2}{n\pi c} \int_0^1 (0) \sin n\pi x \, dx = 0.$$

Evaluate the coefficients A_n now.

$$\begin{aligned}
A_n &= \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx \\
&= 2\left[\int_0^{1/2}(2x)\sin n\pi x\,dx + \int_{1/2}^1 2(1-x)\sin n\pi x\,dx\right] \\
&= 4\left[\int_0^{1/2} x\sin n\pi x\,dx + \int_{1/2}^1 (1-x)\sin n\pi x\,dx\right] \\
&= 4\left(\int_0^{1/2} x\sin n\pi x\,dx + \int_{1/2}^1 \sin n\pi x\,dx - \int_{1/2}^1 x\sin n\pi x\,dx\right) \\
&= 4\left[\int_0^{1/2} \frac{\partial}{\partial k}(-\cos kx)\bigg|_{k=n\pi} dx + \int_{1/2}^1 \sin n\pi x\,dx - \int_{1/2}^1 \frac{\partial}{\partial k}(-\cos kx)\bigg|_{k=n\pi} dx\right] \\
&= 4\left[-\frac{d}{dk}\left(\int_0^{1/2}\cos kx\,dx\right)\bigg|_{k=n\pi} + \int_{1/2}^1 \sin n\pi x\,dx + \frac{d}{dk}\left(\int_{1/2}^1 \cos kx\,dx\right)\bigg|_{k=n\pi}\right] \\
&= 4\left[-\frac{d}{dk}\left(\frac{1}{k}\sin kx\bigg|_0^{1/2}\right)\bigg|_{k=n\pi} + \left(-\frac{1}{n\pi}\cos n\pi x\right)\bigg|_{1/2}^1 + \frac{d}{dk}\left(\frac{1}{k}\sin kx\bigg|_{1/2}^1\right)\bigg|_{k=n\pi}\right] \\
&= 4\left[-\frac{d}{dk}\left(\frac{\sin\frac{k}{2}}{k}\right)\bigg|_{k=n\pi} + \left(\frac{\cos\frac{n\pi}{2} - \cos n\pi}{n\pi}\right) + \frac{d}{dk}\left(\frac{\sin k - \sin\frac{k}{2}}{k}\right)\bigg|_{k=n\pi}\right] \\
&= 4\left[-\left(\frac{k\cos\frac{k}{2} - 2\sin\frac{k}{2}}{2k^2}\right)\bigg|_{k=n\pi} + \left(\frac{\cos\frac{n\pi}{2} - \cos n\pi}{n\pi}\right) + \left(\frac{2k\cos k - k\cos\frac{k}{2} + 2\sin\frac{k}{2} - 2\sin k}{2k^2}\right)\bigg|_{k=n\pi}\right] \\
&= 4\left[-\left(\frac{n\pi\cos\frac{n\pi}{2} - 2\sin\frac{n\pi}{2}}{2n^2\pi^2}\right) + \left(\frac{\cos\frac{n\pi}{2} - \cos n\pi}{n\pi}\right) + \left(\frac{2n\pi\cos n\pi - n\pi\cos\frac{n\pi}{2} + 2\sin\frac{n\pi}{2} - 2\sin n\pi}{2n^2\pi^2}\right)\right] \\
&= 4\left(\frac{\cancel{-n\pi\cos\frac{n\pi}{2}} + 2\sin\frac{n\pi}{2} + \cancel{2n\pi\cos\frac{n\pi}{2}} - \cancel{2n\pi\cos n\pi} + \cancel{2n\pi\cos n\pi} - \cancel{n\pi\cos\frac{n\pi}{2}} + 2\sin\frac{n\pi}{2} - 2\sin n\pi}{2n^2\pi^2}\right) \\
&= 4\left(\frac{4\sin\frac{n\pi}{2} - 2\sin n\pi}{2n^2\pi^2}\right) \\
&= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\underbrace{\sin n\pi}_{=\,0} \\
&= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}
\end{aligned}$$

Consequently, the general solution becomes (with $c = 1$ and $L = 1$)

$$u(x,t) = \sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}\cos n\pi t\sin n\pi x.$$

Notice that because of the $\sin(n\pi/2)$ factor, the summand is zero when n is even. The series can be made to converge faster, then, by summing over the odd integers only. Substitute $n = 2m - 1$.

$$u(x,t) = \sum_{2m-1=1}^{\infty} \frac{8}{(2m-1)^2\pi^2} \overbrace{\sin\frac{(2m-1)\pi}{2}}^{=(-1)^{m-1}} \cos[(2m-1)\pi t] \sin[(2m-1)\pi x]$$

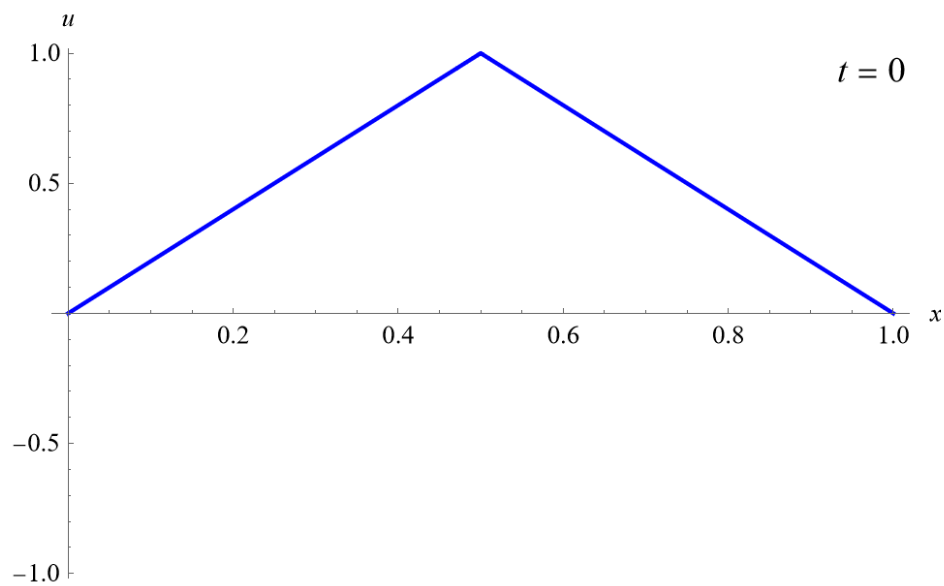
Therefore,

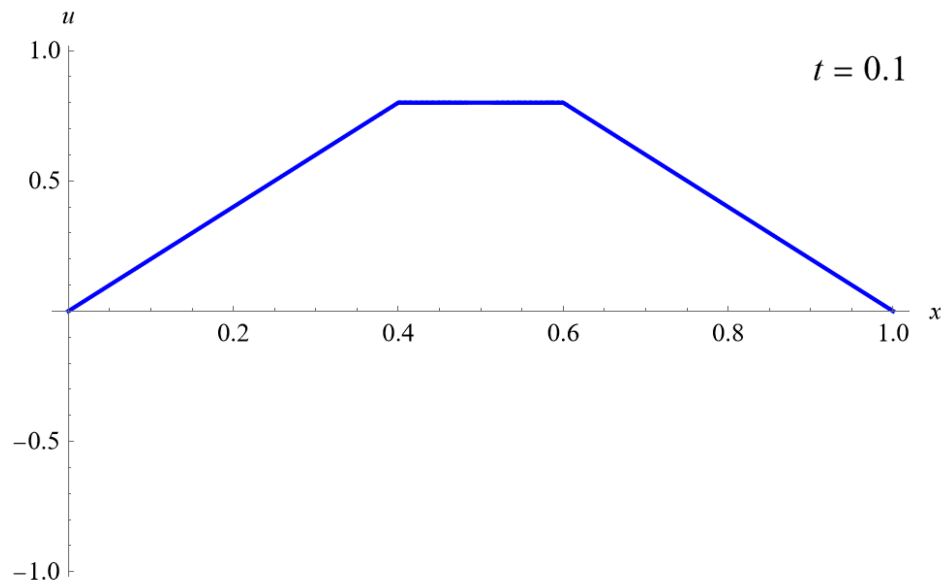
$$u(x,t) = \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)^2} \cos[(2m-1)\pi t] \sin[(2m-1)\pi x].$$

The period of the fundamental ($m = 1$) is

$$\tau = \frac{2\pi}{\omega_1} = \frac{2\pi}{\pi} = 2.$$

Below are 21 plots of $u(x,t)$ versus x at various times from $t = 0$ to $t = \tau$ in order to illustrate the solution's behavior. The first 1000 terms are used in the series for each graph.

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0.2
0.4
0.6
0.8
1.0
x
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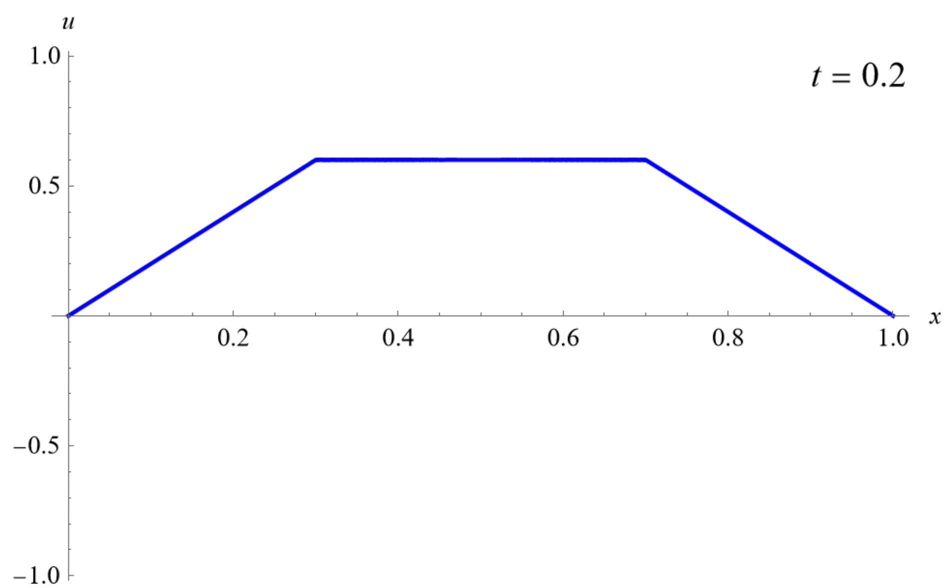
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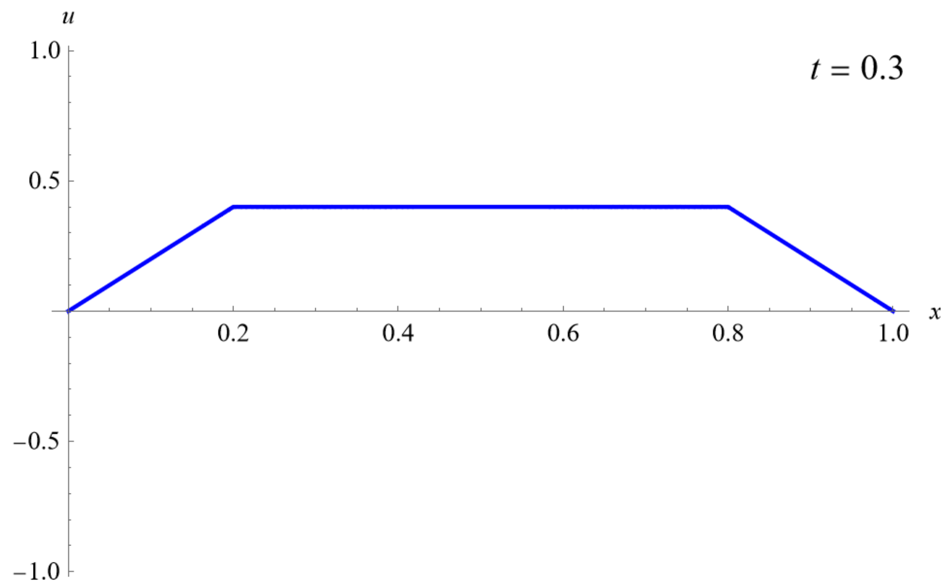
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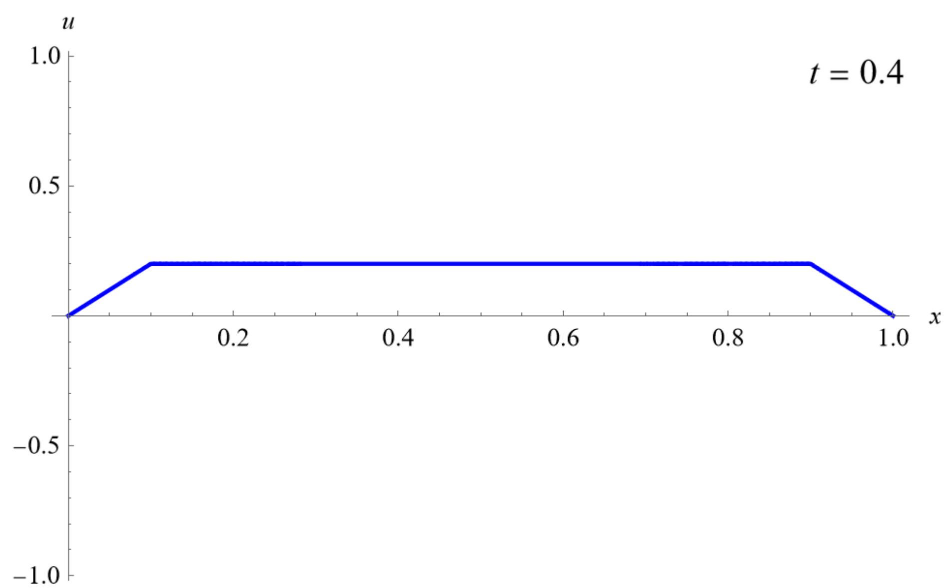
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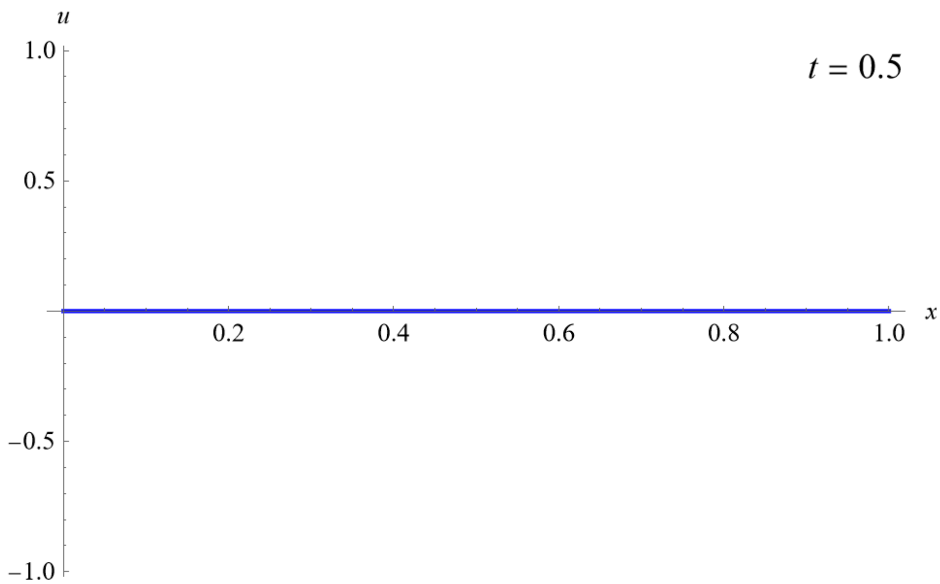
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x
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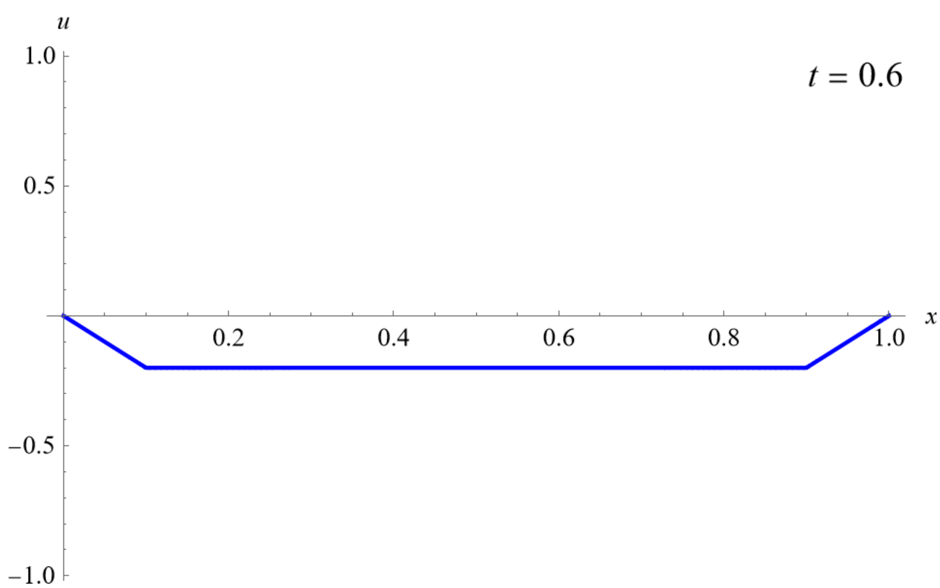
1.0
t = 0.5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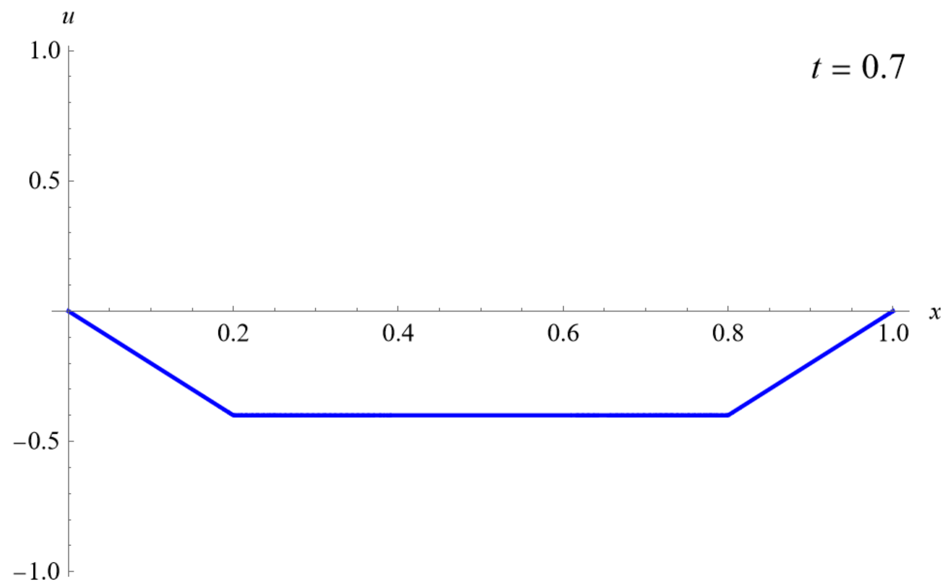
t = 0.6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x
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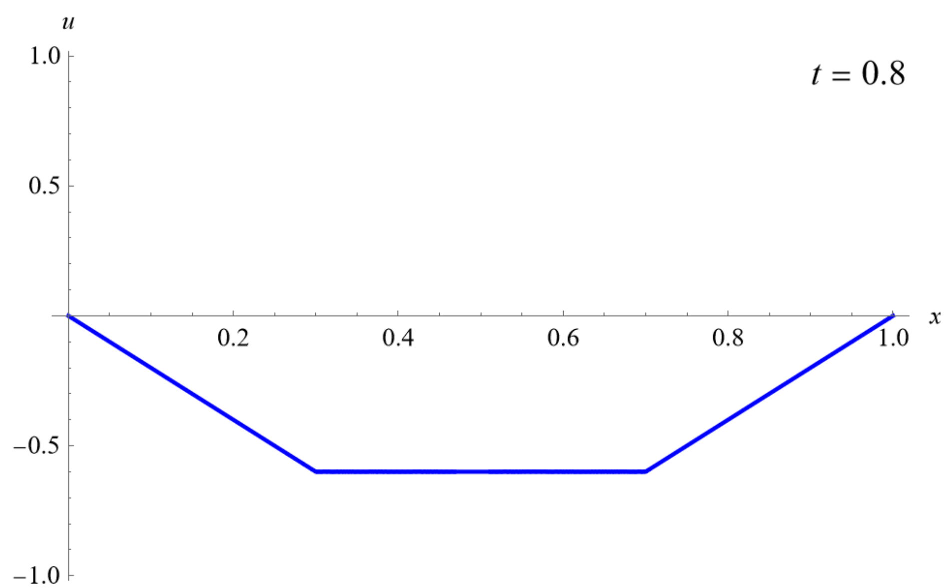
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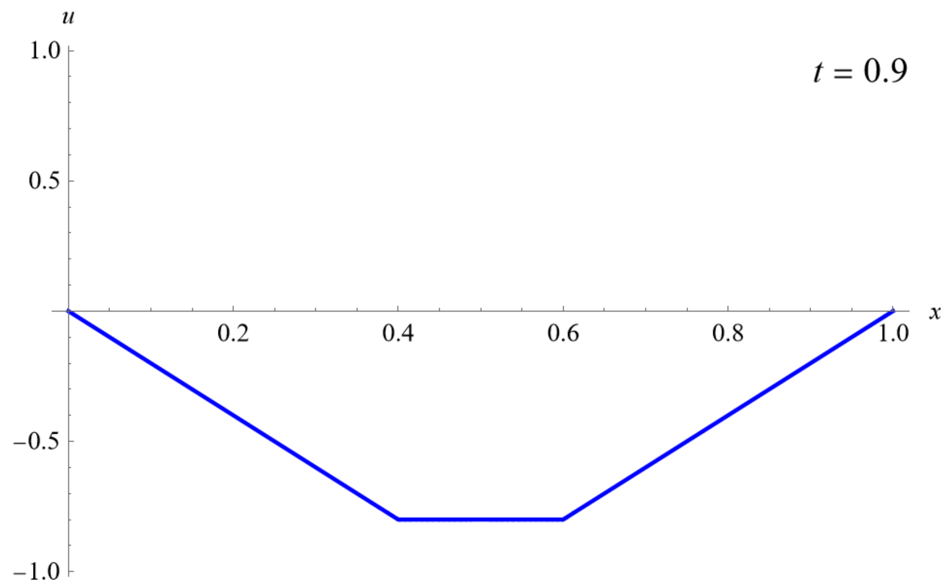
x
t = 0.8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0.6
0.8
1.0
x
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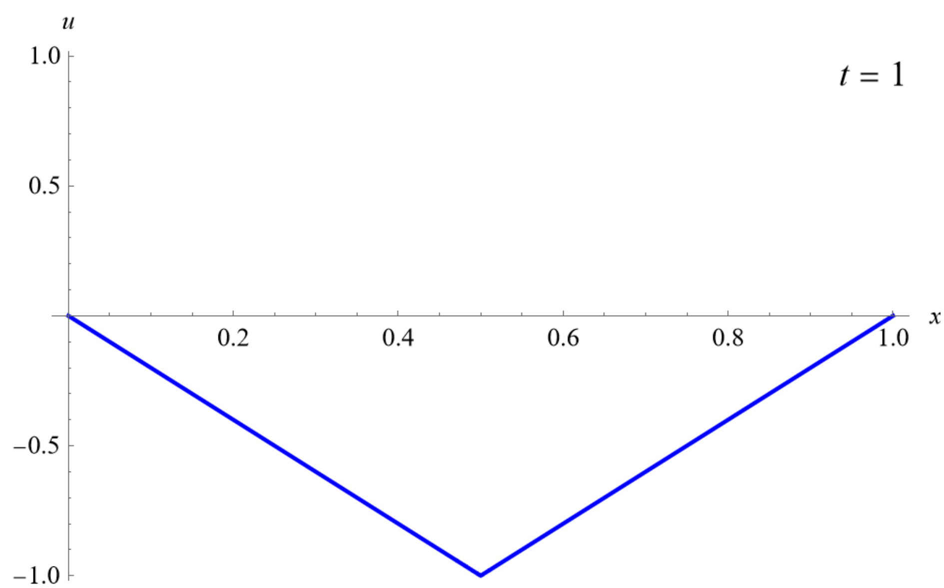
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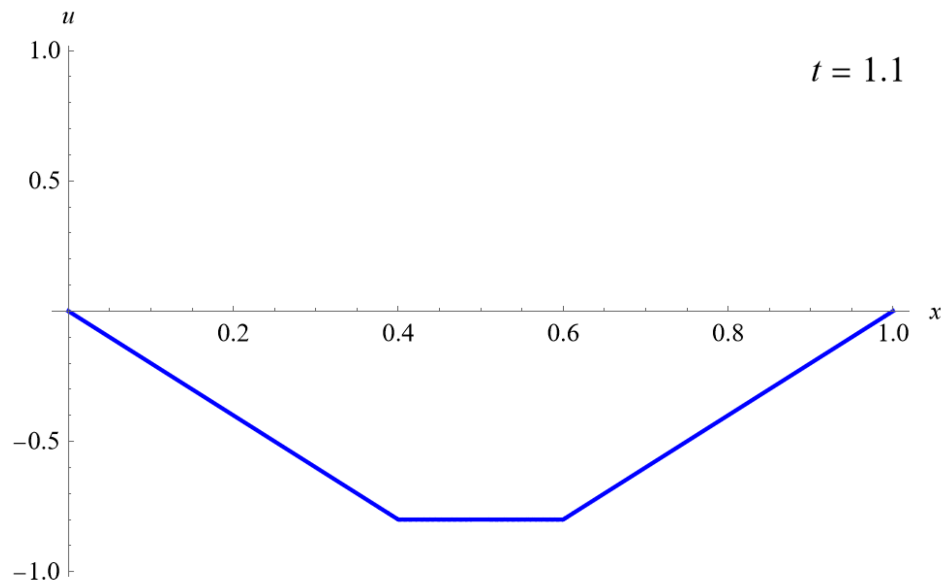
0.8
1.0
x
t = 1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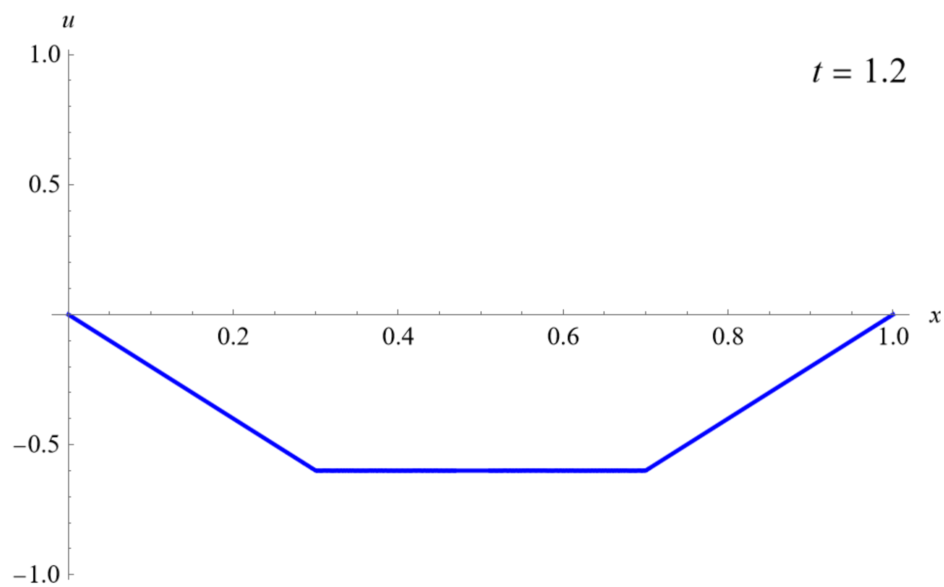
0.4
0.6
0.8
1.0
x
t = 1.1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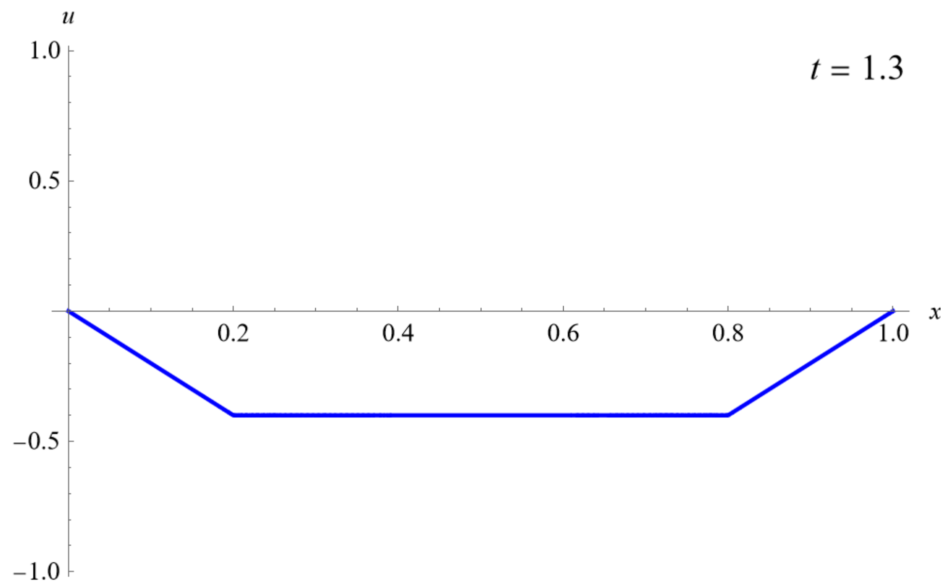
0.4
0.6
0.8
1.0
x
t = 1.2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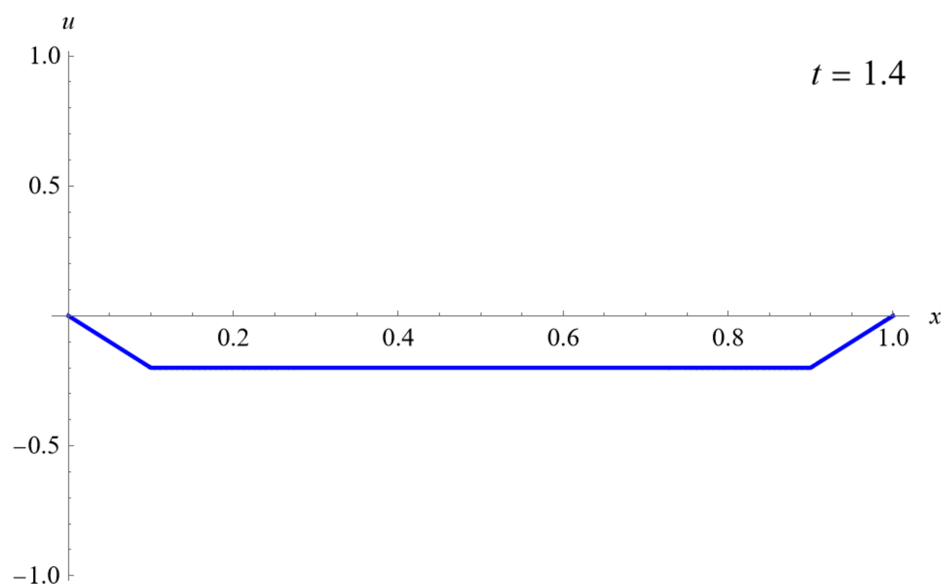
0.6
0.8
1.0
x
t = 1.3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−0.5
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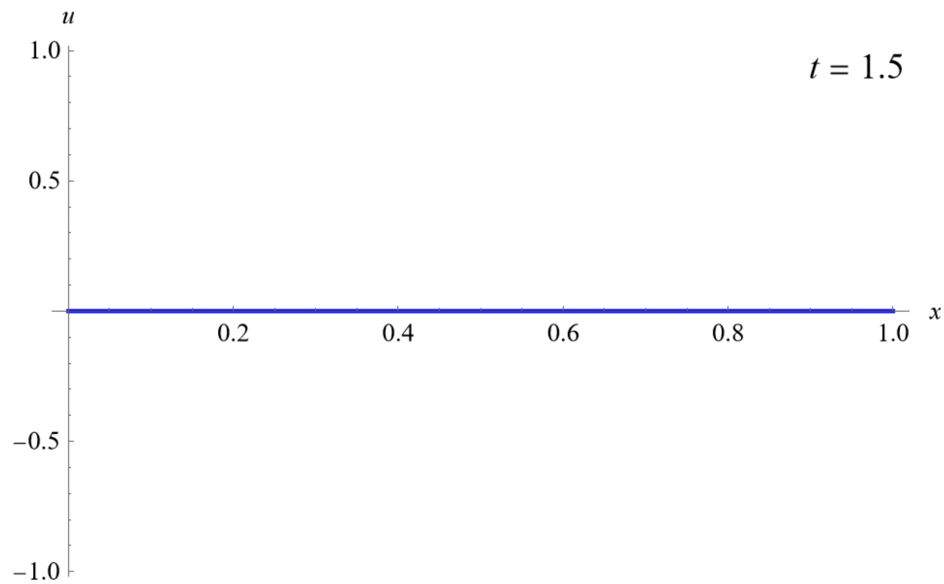
0.2
0.4
0.6
0.8
1.0
x
t = 1.4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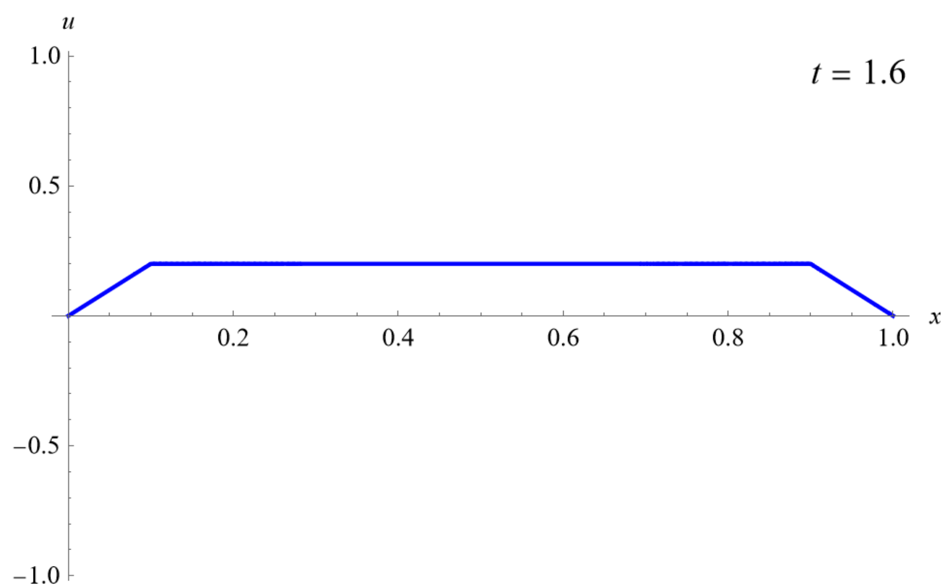
t = 1.5
0.5
x
0.2
0.4
0.6
0.8
1.0
−0.5
−1.0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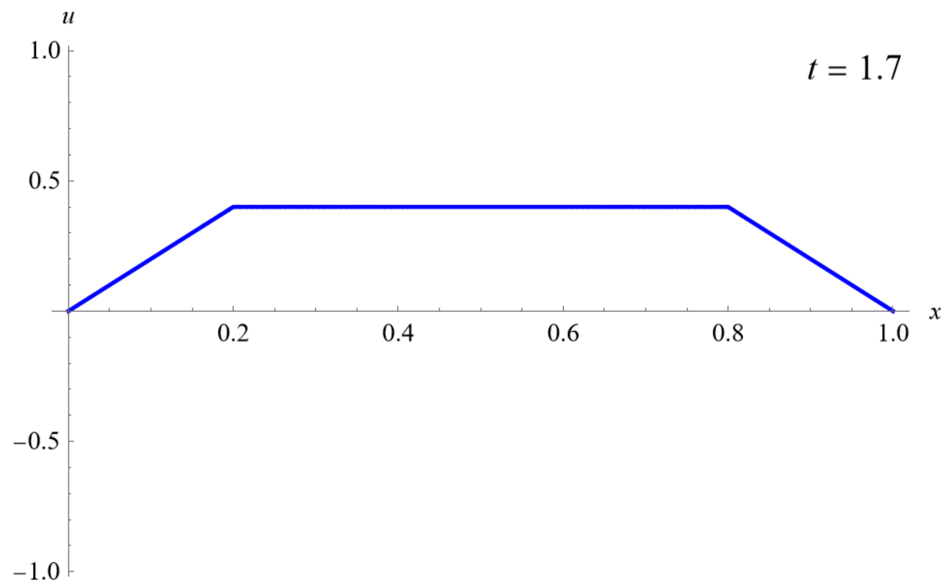
t = 1.6
0.5
x
0.2
0.4
0.6
0.8
1.0
−0.5
−1.0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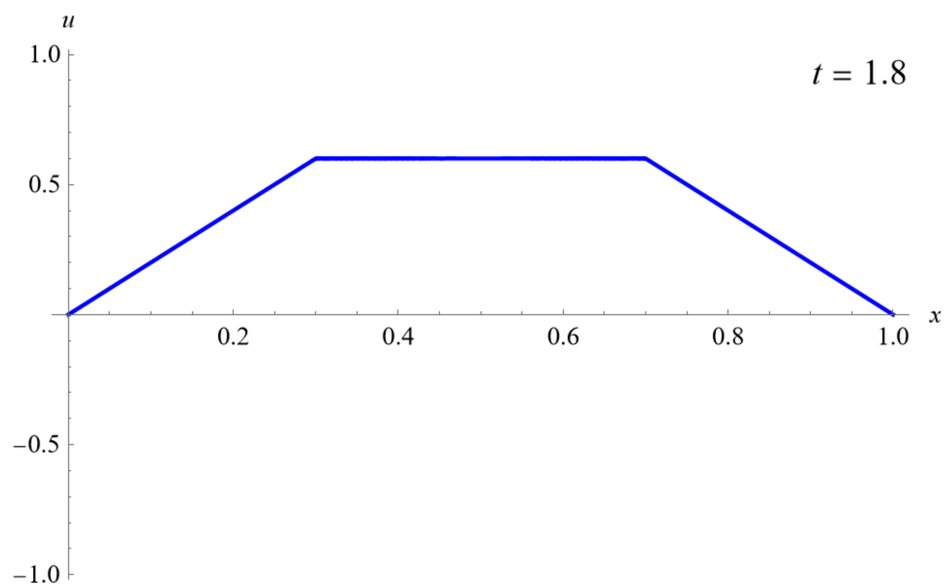
0.6
0.8
1.0
x
t = 1.7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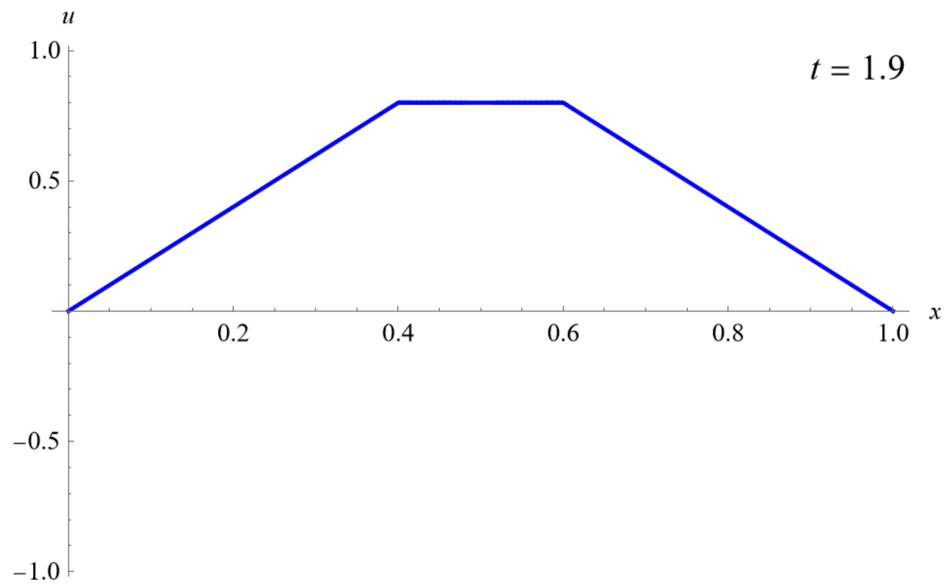
x
t = 1.8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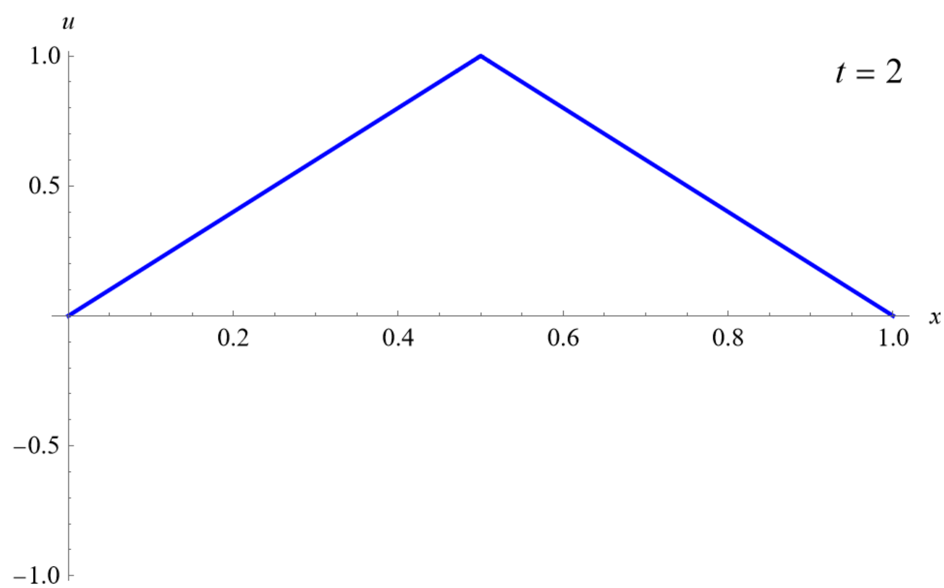
0.6
0.8
1.0
x
t = 1.9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0.8
1.0
x
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