Problem 16.15

Let $f(\xi)$ be any function with first two derivatives $f'(\xi)$ and $f''(\xi)$, and let $\mathbf{n}$ be an arbitrary fixed unit vector. **(a)** Show that $\nabla f(\mathbf{n}\cdot\mathbf{r}-ct)=\mathbf{n}f'(\mathbf{n}\cdot\mathbf{r}-ct)$. **(b)** Hence show that $f(\mathbf{n}\cdot\mathbf{r}-ct)$ satisfies the three-dimensional wave equation (16.38). **(c)** Argue that $f(\mathbf{n}\cdot\mathbf{r}-ct)$ represents a signal that is constant in any plane perpendicular to $\mathbf{n}$ (at any fixed time t) and propagates rigidly with speed c in the direction of $\mathbf{n}$.

[TYPO: The three-dimensional wave equation is in equation (16.39) on page 695.]

Solution

The arbitrary fixed unit vector is $\mathbf{n}=\langle n_x,n_y,n_z\rangle$ with $|\mathbf{n}|=\sqrt{n_x^2+n_y^2+n_z^2}=1$, and $\mathbf{r}=\langle x,y,z\rangle$ is the position vector with $|\mathbf{r}|=\sqrt{x^2+y^2+z^2}=r$.

Part (a)

$$\begin{aligned}
\nabla f(\mathbf{n}\cdot\mathbf{r}-ct) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right) f\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j n_j\right)\cdot\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k x_k\right)-ct\right]\\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right) f\left[\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)n_jx_k-ct\right]\\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right) f\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\delta_{jk}n_jx_k-ct\right)\\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right) f\left(\sum_{j=1}^{3}n_jx_j-ct\right)\\
&= \sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\left[f\left(\sum_{j=1}^{3}n_jx_j-ct\right)\right]\\
&= \sum_{i=1}^{3}\boldsymbol{\delta}_i f'\left(\sum_{j=1}^{3}n_jx_j-ct\right)\cdot\frac{\partial}{\partial x_i}\left(\sum_{l=1}^{3}n_lx_l-ct\right)\\
&= \sum_{i=1}^{3}\boldsymbol{\delta}_i f'\left(\sum_{j=1}^{3}n_jx_j-ct\right)\cdot\left[\frac{\partial}{\partial x_i}\left(\sum_{l=1}^{3}n_lx_l\right)-\frac{\partial}{\partial x_i}(ct)\right]\\
&= \sum_{i=1}^{3}\boldsymbol{\delta}_i f'\left(\sum_{j=1}^{3}n_jx_j-ct\right)\cdot\left[\sum_{l=1}^{3}\frac{\partial}{\partial x_i}(n_lx_l)-0\right]
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\nabla f(\mathbf{n}\cdot\mathbf{r}-ct) &= \sum_{i=1}^{3}\boldsymbol{\delta}_i f'\left(\sum_{j=1}^{3}n_j x_j - ct\right)\cdot\left[\sum_{l=1}^{3}n_l\frac{\partial}{\partial x_i}(x_l)\right] \\
&= \sum_{i=1}^{3}\boldsymbol{\delta}_i f'\left(\sum_{j=1}^{3}n_j x_j - ct\right)\cdot\left(\sum_{l=1}^{3}n_l\delta_{il}\right) \\
&= \sum_{i=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_i n_l\delta_{il} f'\left(\sum_{j=1}^{3}n_j x_j - ct\right) \\
&= \sum_{i=1}^{3}\boldsymbol{\delta}_i n_i f'\left(\sum_{j=1}^{3}n_j x_j - ct\right) \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i n_i\right) f'\left(\sum_{j=1}^{3}n_j x_j - ct\right) \\
&= \mathbf{n} f'(\mathbf{n}\cdot\mathbf{r}-ct)
\end{aligned}$$

Part (b)

The three-dimensional wave equation is given by

$$\frac{\partial^2 f}{\partial t^2} = c^2\nabla^2 f.$$

Check to see whether or not $f(\mathbf{n}\cdot\mathbf{r}-ct)$ satisfies this equation by evaluating both sides.

$$\begin{aligned}
\frac{\partial}{\partial t}f(\mathbf{n}\cdot\mathbf{r}-ct) &= f'(\mathbf{n}\cdot\mathbf{r}-ct)\cdot\frac{\partial}{\partial t}(\mathbf{n}\cdot\mathbf{r}-ct) \\
&= f'(\mathbf{n}\cdot\mathbf{r}-ct)\cdot(-c) \\
&= -cf'(\mathbf{n}\cdot\mathbf{r}-ct)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}f(\mathbf{n}\cdot\mathbf{r}-ct) &= \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}f(\mathbf{n}\cdot\mathbf{r}-ct)\right] \\
&= \frac{\partial}{\partial t}\left[-cf'(\mathbf{n}\cdot\mathbf{r}-ct)\right] \\
&= -cf''(\mathbf{n}\cdot\mathbf{r}-ct)\cdot\frac{\partial}{\partial t}(\mathbf{n}\cdot\mathbf{r}-ct) \\
&= -cf''(\mathbf{n}\cdot\mathbf{r}-ct)\cdot(-c) \\
&= c^2 f''(\mathbf{n}\cdot\mathbf{r}-ct)
\end{aligned}$$

Now evaluate the right side.

$$
\begin{aligned}
c^2\nabla^2 f &= c^2(\nabla\cdot\nabla)f(\mathbf{n}\cdot\mathbf{r}-ct) \\
&= c^2\left[\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\right]f(\mathbf{n}\cdot\mathbf{r}-ct) \\
&= c^2\left[\sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right]f(\mathbf{n}\cdot\mathbf{r}-ct) \\
&= c^2\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right)f(\mathbf{n}\cdot\mathbf{r}-ct) \\
&= c^2\left(\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\right)f\left(\sum_{k=1}^{3}n_kx_k-ct\right) \\
&= c^2\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_i}f\left(\sum_{k=1}^{3}n_kx_k-ct\right)\right] \\
&= c^2\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[f'\left(\sum_{k=1}^{3}n_kx_k-ct\right)\cdot\frac{\partial}{\partial x_i}\left(\sum_{k=1}^{3}n_kx_k-ct\right)\right] \\
&= c^2\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[f'\left(\sum_{k=1}^{3}n_kx_k-ct\right)\cdot\left(\sum_{k=1}^{3}n_k\frac{\partial}{\partial x_i}x_k\right)\right] \\
&= c^2\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[f'\left(\sum_{k=1}^{3}n_kx_k-ct\right)\cdot\left(\sum_{k=1}^{3}n_k\delta_{ik}\right)\right] \\
&= c^2\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[f'\left(\sum_{k=1}^{3}n_kx_k-ct\right)\cdot(n_i)\right] \\
&= c^2\sum_{i=1}^{3}n_i\frac{\partial}{\partial x_i}f'\left(\sum_{k=1}^{3}n_kx_k-ct\right) \\
&= c^2\sum_{i=1}^{3}n_if''\left(\sum_{k=1}^{3}n_kx_k-ct\right)\cdot\frac{\partial}{\partial x_i}\left(\sum_{k=1}^{3}n_kx_k-ct\right) \\
&= c^2\sum_{i=1}^{3}n_if''\left(\sum_{k=1}^{3}n_kx_k-ct\right)\cdot(n_i) \\
&= c^2\sum_{i=1}^{3}n_i^2f''\left(\sum_{k=1}^{3}n_kx_k-ct\right)
\end{aligned}
$$

Continue the simplification.

$$\begin{aligned} c^2\nabla^2 f &= c^2\left(\sum_{i=1}^3 n_i^2\right) f''\left(\sum_{k=1}^3 n_k x_k - ct\right) \\ &= c^2(1)f''\left(\sum_{k=1}^3 n_k x_k - ct\right) \\ &= c^2 f''(\mathbf{n}\cdot\mathbf{r} - ct) \end{aligned}$$

Therefore, since

$$\frac{\partial^2 f}{\partial t^2} = c^2\nabla^2 f,$$

$f(\mathbf{n}\cdot\mathbf{r} - ct)$ satisfies the three-dimensional wave equation.

Part (c)

Because $f(\mathbf{n}\cdot\mathbf{r} - ct)$ satisfies the three-dimensional wave equation, $f(\mathbf{n}\cdot\mathbf{r} - ct)$ is a waveform, or signal. The result of part (a) is

$$\nabla f(\mathbf{n}\cdot\mathbf{r} - ct) = \mathbf{n}f'(\mathbf{n}\cdot\mathbf{r} - ct).$$

The gradient of a function points in the direction of maximum increase. Since the direction of ∇f is $\mathbf{n}$, f is constant in any plane perpendicular to $\mathbf{n}$ (at any fixed time t). In addition, because of the $-ct$ in the argument, the signal moves with speed c in the positive $\mathbf{n}$ direction.