Problem 16.2

The wave equation (16.4) is the equation of motion for a continuous string, as illustrated in Figure 16.1(a). You can obtain this equation as the limit as $n \to \infty$ of the equations for the n discrete masses of Figure 16.1(b). You need to be careful with the limiting process. As $n \to \infty$, the spacing b between the masses (see Figure 16.20) and the individual masses m must both go to zero in such a way that the linear mass density m/b approaches μ, the density of the continuous string. You can guarantee this by taking $m = \mu b$. Write down Newton's second law for the position u_i of the ith mass and show that it goes over to the wave equation as $b \to 0$.

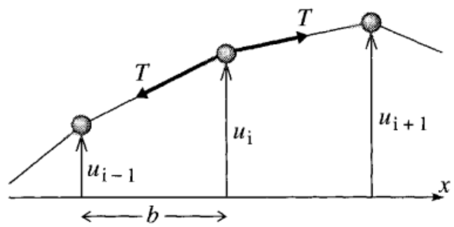
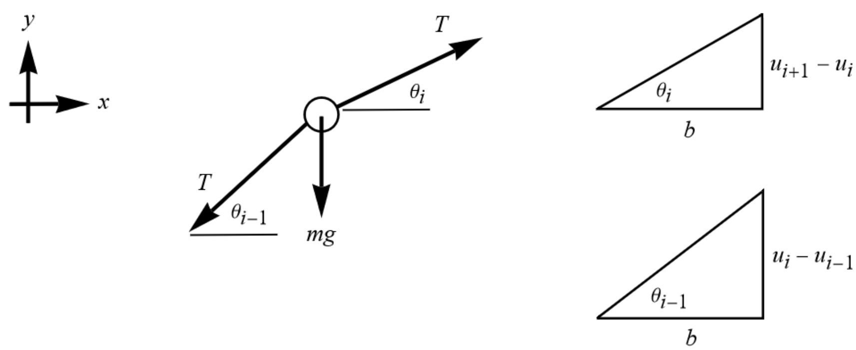


Figure 16.20 Problem 16.2

Solution

Start by drawing a free-body diagram for the ith mass.



Apply Newton's second law to the ith mass.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \left.\begin{aligned} T\cos\theta_i - T\cos\theta_{i-1} &= ma_x \\ T\sin\theta_i - T\sin\theta_{i-1} - mg &= ma_y \end{aligned}\right\}$$

The motion is assumed to be entirely vertical, so $a_x = 0$ and $a_y = \partial^2 u_i/\partial t^2$.

$$\left.\begin{aligned} T\cos\theta_i - T\cos\theta_{i-1} &= 0 \\ T\sin\theta_i - T\sin\theta_{i-1} - mg &= m\frac{\partial^2 u_i}{\partial t^2} \end{aligned}\right\}$$

As a result, the first equation doesn't tell us anything.

$$T\sin\theta_i - T\sin\theta_{i-1} - mg = m\frac{\partial^2 u_i}{\partial t^2}$$

Assume that θ_i and θ_{i-1} are small so that $\sin\theta_i \approx \tan\theta_i$ and $\sin\theta_{i-1} \approx \tan\theta_{i-1}$, respectively.

$$T\tan\theta_i - T\tan\theta_{i-1} - mg = m\frac{\partial^2 u_i}{\partial t^2}$$

Substitute formulas for the tangents.

$$T\left(\frac{u_{i+1}-u_i}{b}\right) - T\left(\frac{u_i - u_{i-1}}{b}\right) - mg = m\frac{\partial^2 u_i}{\partial t^2}$$

Rewrite the left side.

$$T\left(\frac{\frac{u_{i+1}-u_i}{b} - \frac{u_i-u_{i-1}}{b}}{b}\right)b - mg = m\frac{\partial^2 u_i}{\partial t^2} \tag{1}$$

The two fractions in the numerator are forward and backward difference quotients.

$$\frac{\Delta u}{\Delta x} = \frac{u_{i+1}-u_i}{b} \quad \text{or} \quad \frac{u_i-u_{i-1}}{b}$$

Consequently, the entire quantity in parentheses approaches the value of $\partial^2 u_i/\partial x^2$ in the limit as $b \to 0$. Divide both sides of equation (1) by b.

$$T\left(\frac{\frac{u_{i+1}-u_i}{b} - \frac{u_i-u_{i-1}}{b}}{b}\right) - \frac{m}{b}g = \frac{m}{b}\frac{\partial^2 u_i}{\partial t^2}$$

Take the limit as $b \to 0$.

$$T\frac{\partial^2 u}{\partial x^2} - \mu g = \mu\frac{\partial^2 u}{\partial t^2} \tag{2}$$

The individual masses now form a string, and u_i is now $u(x,t)$. If the tension in this string is dominant over the mass density, that is,

$$T\frac{\partial^2 u}{\partial x^2} \gg \mu g,$$

then the term μg can be neglected, resulting in the wave equation.

$$T\frac{\partial^2 u}{\partial x^2} = \mu\frac{\partial^2 u}{\partial t^2}$$

If the tension is not dominant over the density, then take advantage of the fact that equation (2) is linear by writing the general solution as the sum of an equilibrium component and a transient component: $u(x,t) = u_E(x) + v(x,t)$.

$$T\frac{\partial^2}{\partial x^2}[u_E(x) + v(x,t)] - \mu g = \mu\frac{\partial^2}{\partial t^2}[u_E(x) + v(x,t)]$$

$$T\left(\frac{d^2 u_E}{dx^2} + \frac{\partial^2 v}{\partial x^2}\right) - \mu g = \mu\frac{\partial^2 v}{\partial t^2}$$

$$T\frac{d^2 u_E}{dx^2} + T\frac{\partial^2 v}{\partial x^2} - \mu g = \mu\frac{\partial^2 v}{\partial t^2}$$

If we set

$$T\frac{d^2 u_E}{dx^2} - \mu g = 0 \qquad \Rightarrow \qquad u_E(x) = \frac{\mu g}{2T}x^2 + C_1 x + C_2,$$

then the transient component satisfies the wave equation.

$$T\frac{\partial^2 v}{\partial x^2} = \mu\frac{\partial^2 v}{\partial t^2}$$

What this means is the string will oscillate about a sagged (parabolic) equilibrium position $y = u_E(x)$ rather than $y = 0$.