Problem 16.3

Let $f(\xi)$ be an arbitrary (twice differentiable) function. Show by direct substitution that $f(x-ct)$ is a solution of the wave equation (16.4).

Solution

The wave equation is given by equation (16.4) on page 684.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{16.4}$$

Find the derivatives of the given function $u(x,t) = f(x-ct)$ by using the chain rule.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} f(x-ct) = f'(x-ct)\frac{\partial}{\partial t}(x-ct) = f'(x-ct)(-c) = -cf'(x-ct)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left[-cf'(x-ct)\right] = -cf''(x-ct)\frac{\partial}{\partial t}(x-ct) = -cf''(x-ct)(-c) = c^2 f''(x-ct)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(x-ct) = f'(x-ct)\frac{\partial}{\partial x}(x-ct) = f'(x-ct)(1) = f'(x-ct)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x} f'(x-ct) = f''(x-ct)\frac{\partial}{\partial x}(x-ct) = f''(x-ct)(1) = f''(x-ct)$$

Notice that

$$\frac{\partial^2 u}{\partial t^2} = c^2 f''(x-ct) = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Therefore, $u(x,t) = f(x-ct)$ is a solution of the wave equation.