Problem 16.6

There is a small flaw in Example 16.1 (page 686). In Equation (16.14) I omitted a constant of integration, so the equation should really have read $f(x) - g(x) = k$. Show that this still gives the same final answer (16.15).

Solution

The aim is to solve the wave equation for a string with a certain shape initially that's released from rest. Skip to page 3 to get on with the problem.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \; -\infty < t < \infty$$

$$u(x,0) = u_0(x)$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

Because it needs to be solved over the whole line ($-\infty < x < \infty$), the method of operator factorization can be applied here. Bring both terms to the left side.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Write the left side as an operator applied to u.

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = 0$$

The operator is a difference of squares, so factor it.

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0 \tag{1}$$

If we set

$$v = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u,$$

then equation (1) becomes

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) v = 0.$$

As a result of using the method of operator factorization, the wave equation has reduced to a system of first-order PDEs, which can each be solved with the method of characteristics.

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} &= v \\ \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} &= 0 \end{aligned}\right\}$$

Since $v = v(x,t)$ is a two-dimensional function, it's differential is defined as

$$dv = \frac{\partial v}{\partial t}\,dt + \frac{\partial v}{\partial x}\,dx.$$

Dividing both sides by dt yields the fundamental relationship between the total derivative of v with respect to t and its partial derivatives.

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{dx}{dt}\frac{\partial v}{\partial x}$$

Along the characteristic curves in the tx-plane defined by

$$\frac{dx}{dt} = c, \quad x(0) = \xi, \tag{2}$$

where ξ is a characteristic coordinate, the PDE for v reduces to an ODE.

$$\frac{dv}{dt} = 0 \tag{3}$$

Integrate both sides of equation (2) with respect to t.

$$x = ct + \xi \quad \rightarrow \quad \xi = x - ct$$

Now solve for v by integrating both sides of equation (3) with respect to t.

$$v(\xi, t) = f(\xi)$$

Here f is an arbitrary function of ξ. Now that v is known, change back to the original variables.

$$v(x, t) = f(x - ct)$$

Consequently, the PDE for u is

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = f(x - ct).$$

Similar to v, the relationship between the total derivative of u with respect to t and its partial derivatives is

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}.$$

Along the characteristic curves in the tx-plane defined by

$$\frac{dx}{dt} = -c, \quad x(0) = \eta, \tag{4}$$

where η is another characteristic coordinate, the PDE for u reduces to an ODE.

$$\frac{du}{dt} = f(x - ct) \tag{5}$$

Integrate both sides of equation (4) with respect to t.

$$x = -ct + \eta \quad \rightarrow \quad \eta = x + ct$$

Use this result to eliminate x from equation (5).

$$\frac{du}{dt} = f(-2ct + \eta)$$

Solve for u by integrating both sides with respect to t.

$$u(\eta, t) = \int^t f(-2cs + \eta)\, ds + G(\eta)$$

Here G is another arbitrary function. Evaluate the integral, noting that the integral of an arbitrary function is another arbitrary function.

$$u(\eta, t) = F(-2ct + \eta) + G(\eta)$$

Now that u is known, change back to the original variables.

$$u(x, t) = F(x - ct) + G(x + ct)$$

The general solution to the wave equation over the whole line is the sum of a wave travelling to the right and a wave travelling to the left. Differentiate it with respect to t.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} F(x - ct) + \frac{\partial}{\partial t} G(x + ct) \\
&= F'(x - ct)\frac{\partial}{\partial t}(x - ct) + G'(x + ct)\frac{\partial}{\partial t}(x + ct) \\
&= F'(x - ct)(-c) + G'(x + ct)(c) \\
&= -cF'(x - ct) + cG'(x + ct)
\end{aligned}$$

Apply the initial conditions now to determine F and G.

$$u(x, 0) = F(x) + G(x) = u_0(x)$$

$$\frac{\partial u}{\partial t}(x, 0) = -cF'(x) + cG'(x) = 0$$

Differentiate both sides of the first equation, and divide both sides of the second equation by c.

$$F'(x) + G'(x) = u_0'(x) \tag{6}$$

$$-F'(x) + G'(x) = 0 \tag{7}$$

Add the respective sides of equations (6) and (7) to eliminate $F'(x)$.

$$2G'(x) = u_0'(x)$$

Divide both sides by 2.

$$G'(x) = \frac{1}{2} u_0'(x)$$

Integrate both sides with respect to x.

$$G(x) = \frac{1}{2} u_0(x) + C_1$$

Subtract the respective sides of equations (6) and (7) to eliminate $G'(x)$ instead.

$$2F'(x) = u_0'(x)$$

Divide both sides by 2.

$$F'(x) = \frac{1}{2} u_0'(x)$$

Integrate both sides with respect to x.

$$F(x) = \frac{1}{2}u_0(x) + C_2$$

These formulas for $F(x)$ and $G(x)$ are really formulas for $F(w)$ and $G(w)$, where w is any expression we wish.

$$F(w) = \frac{1}{2}u_0(w) + C_2$$

$$G(w) = \frac{1}{2}u_0(w) + C_1$$

In the formula for $u(x,t)$, we need $F(x-ct)$ and $G(x+ct)$.

$$F(x-ct) = \frac{1}{2}u_0(x-ct) + C_2$$

$$G(x+ct) = \frac{1}{2}u_0(x+ct) + C_1$$

So then

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \\ &= \left[\frac{1}{2}u_0(x-ct) + C_2\right] + \left[\frac{1}{2}u_0(x+ct) + C_1\right] \\ &= \frac{1}{2}[u_0(x-ct) + u_0(x+ct)] + C_1 + C_2. \end{aligned}$$

In order for $u(x,0) = u_0(x)$ to be satisfied, it's necessary to set $C_1 + C_2 = 0$. Therefore,

$$u(x,t) = \frac{1}{2}[u_0(x-ct) + u_0(x+ct)].$$

What this means is the initial waveform splits into two waves with half the amplitude, one moving to the right with speed c and the other moving to the left with speed c.