Problem 16.8

[Computer] Make plots similar to Figure 16.5 of the standing wave (16.18) for several equally spaced times from $t = 0$ to τ, the period. Take $2A = 1$ and $k = \omega = 2\pi$. Animate your pictures and describe the motion.

Solution

The general solution to the initial value problem,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ -\infty < t < \infty \\
&u(x, 0) = u_0(x) \\
&\frac{\partial u}{\partial t}(x, 0) = 0,
\end{aligned}$$

was found in Problem 16.6 to be

$$u(x, t) = \frac{1}{2}[u_0(x - ct) + u_0(x + ct)].$$

In the special case that $u_0(x) = A \sin kx$,

$$\begin{aligned}
u(x, t) &= \frac{1}{2}[A \sin k(x - ct) + A \sin k(x + ct)] \\
&= \frac{A}{2}[\sin(kx - \omega t) + \sin(kx + \omega t)] \\
&= \frac{A}{2}\left[2 \sin \frac{(kx - \omega t) + (kx + \omega t)}{2} \cos \frac{(kx - \omega t) - (kx + \omega t)}{2}\right] \\
&= A \sin kx \cos \omega t,
\end{aligned}$$

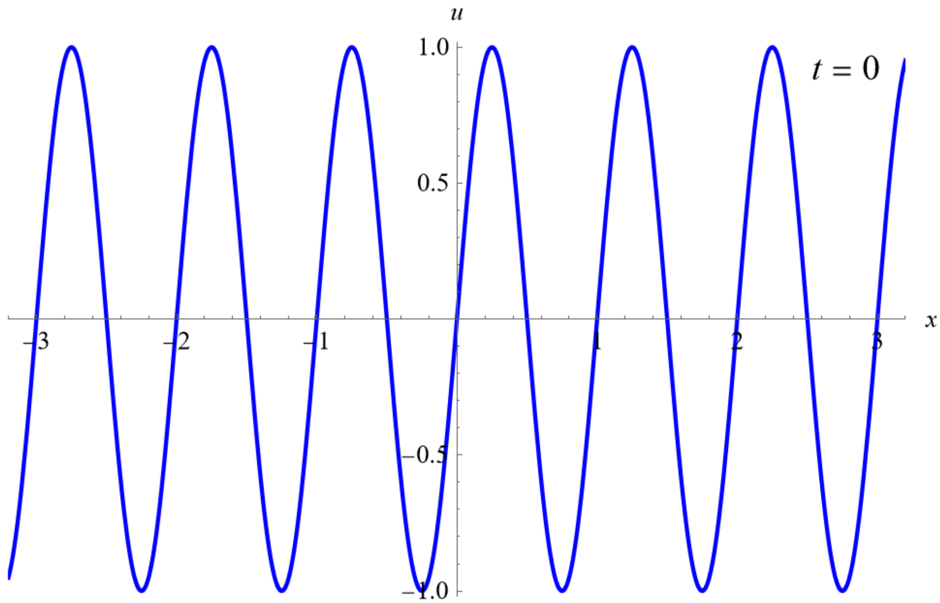
where $\omega = kc$. If $A = 1$ and $k = 2\pi$ and $\omega = 2\pi$, then

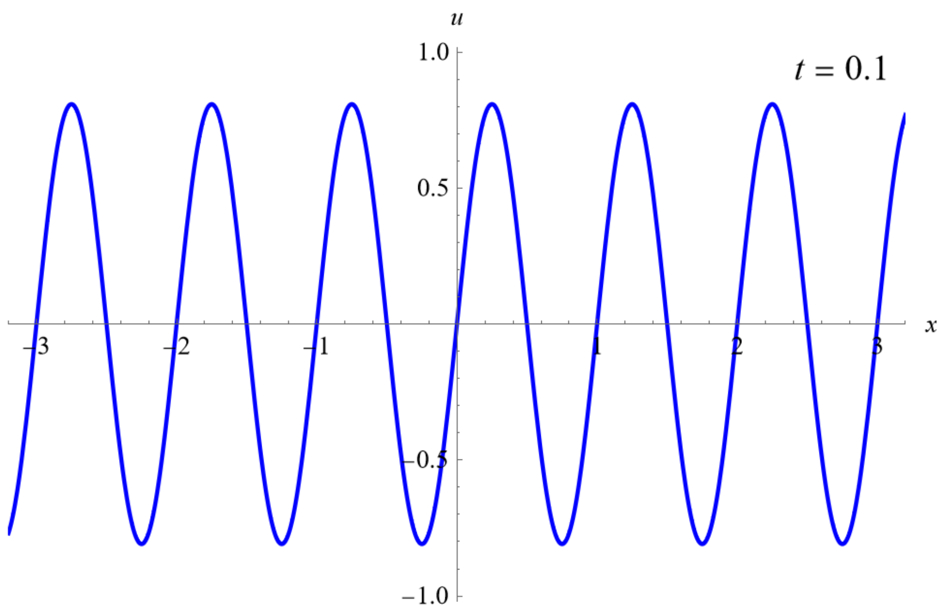
$$u(x, t) = \sin 2\pi x \cos 2\pi t.$$

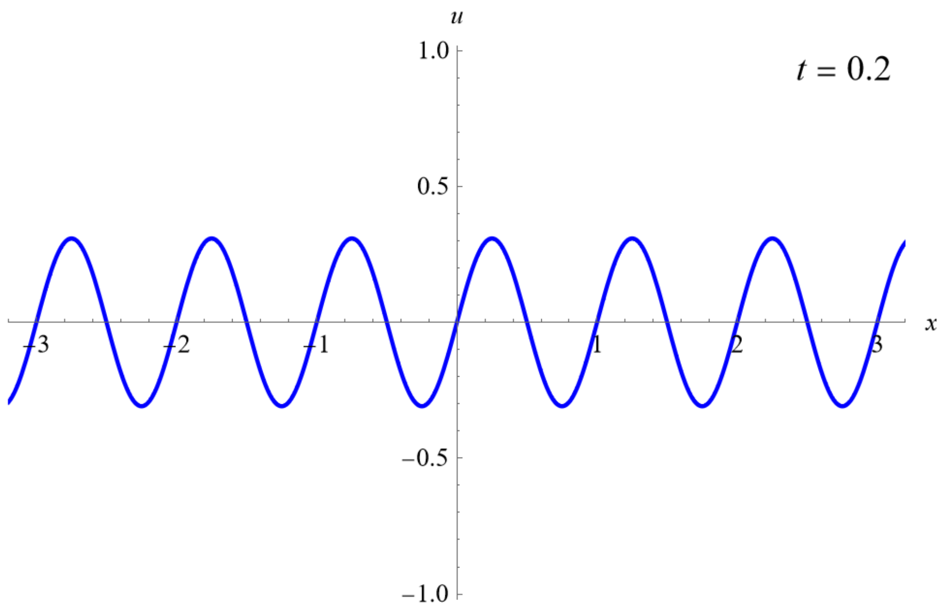
The period is

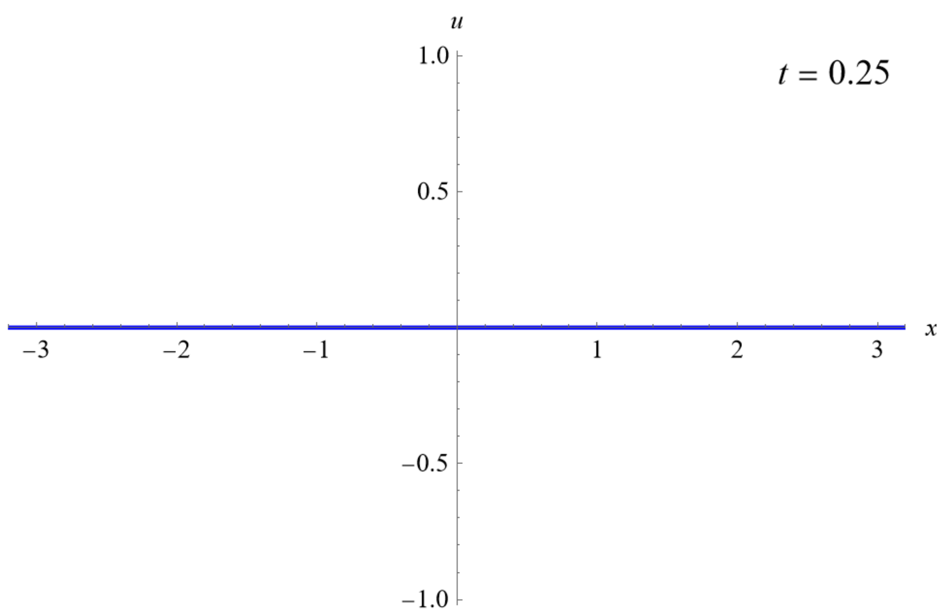
$$\tau = \frac{2\pi}{\omega} = 1.$$

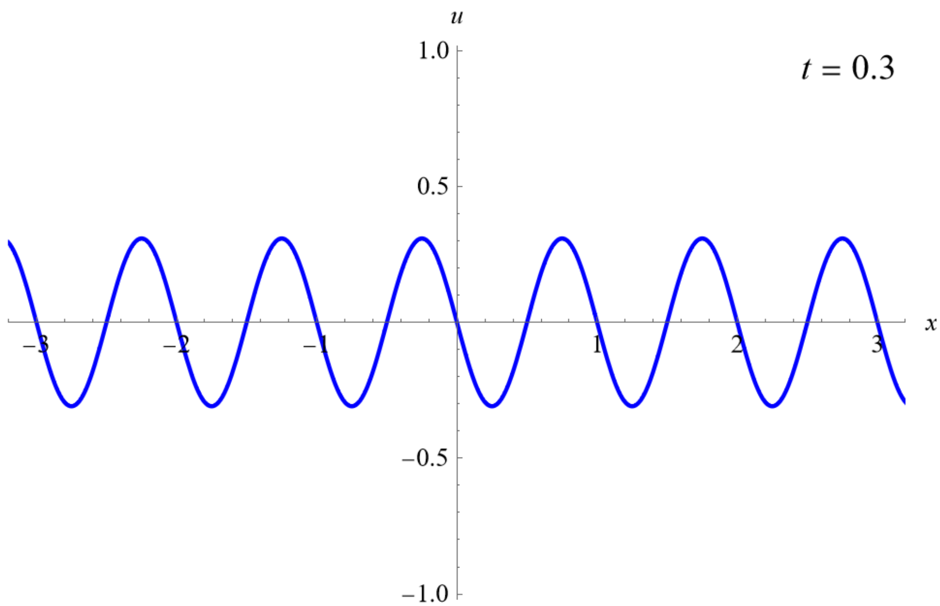
Below are computer-generated plots of $u(x, t)$ versus x at many values of t between 0 and 1.

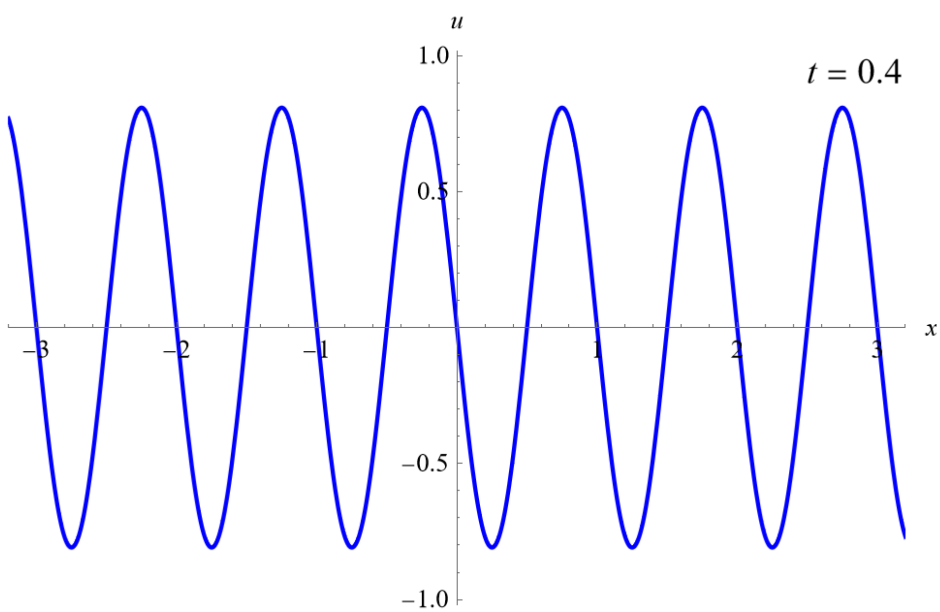


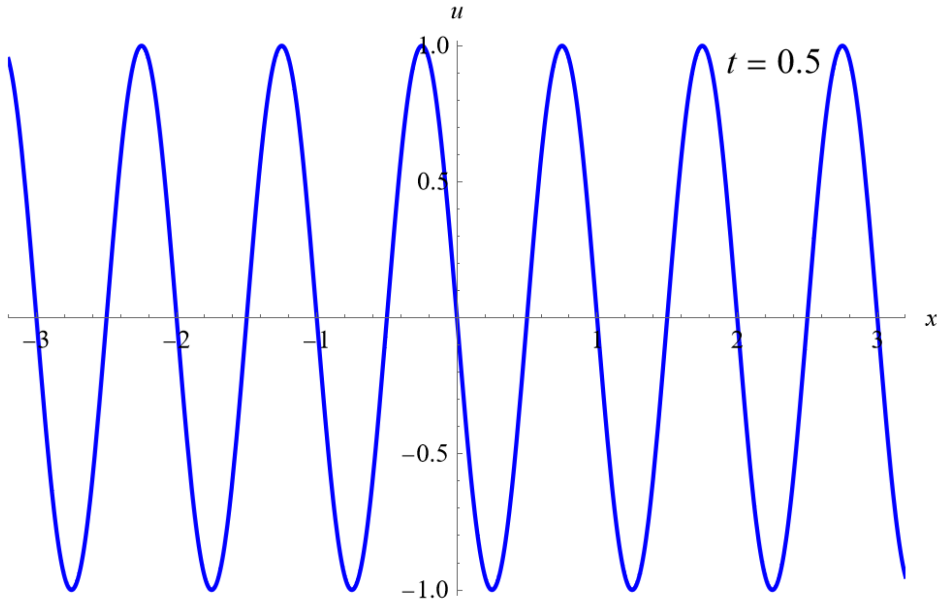


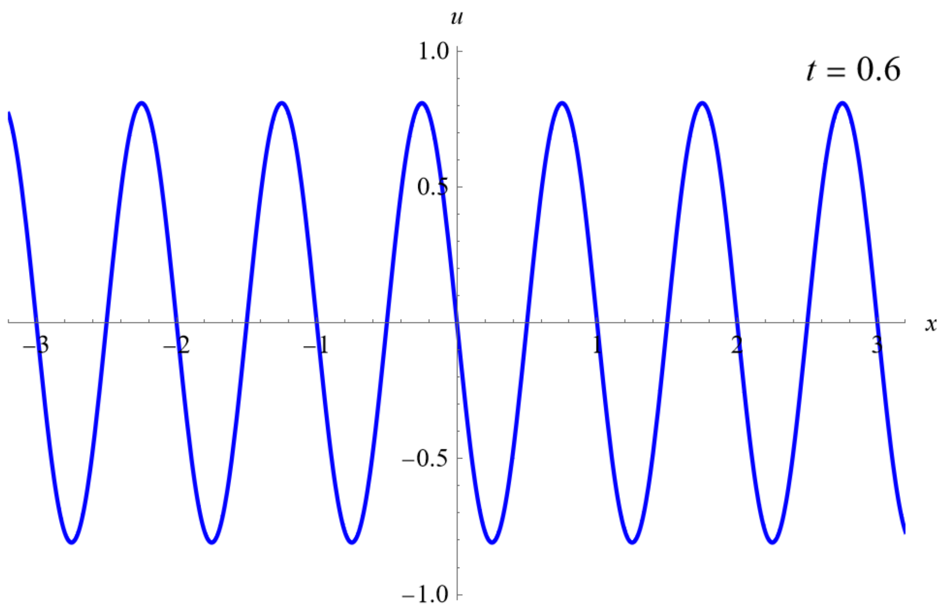

u
1.0
t = 0.2
0.5
x
−3
−2
−1
1
2
3
−0.5
−1.0

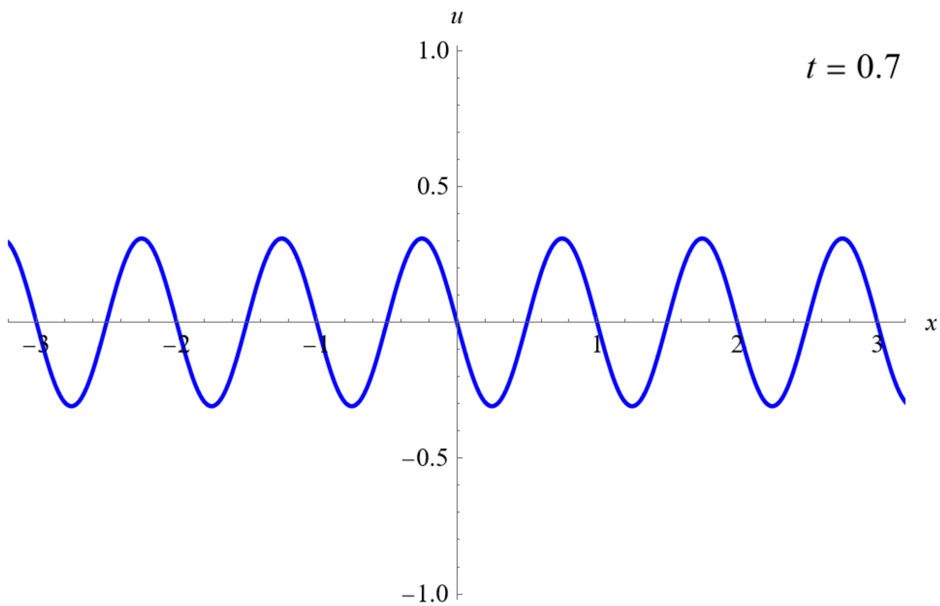

u
1.0
t = 0.25
0.5
x
−3
−2
−1
1
2
3
−0.5
−1.0

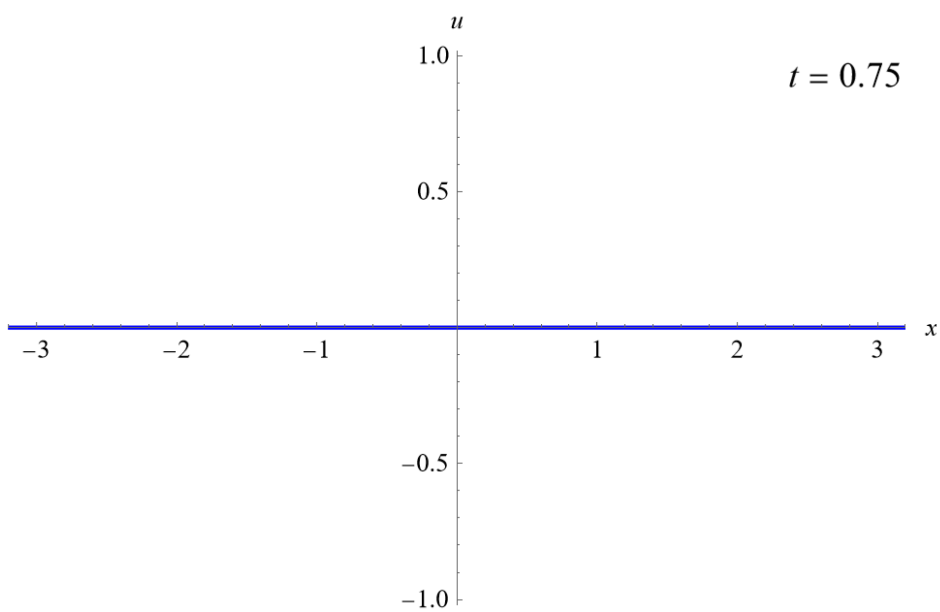

u
1.0
0.5
−0.5
−1.0
−3
−2
−1
1
2
3
x
t = 0.3

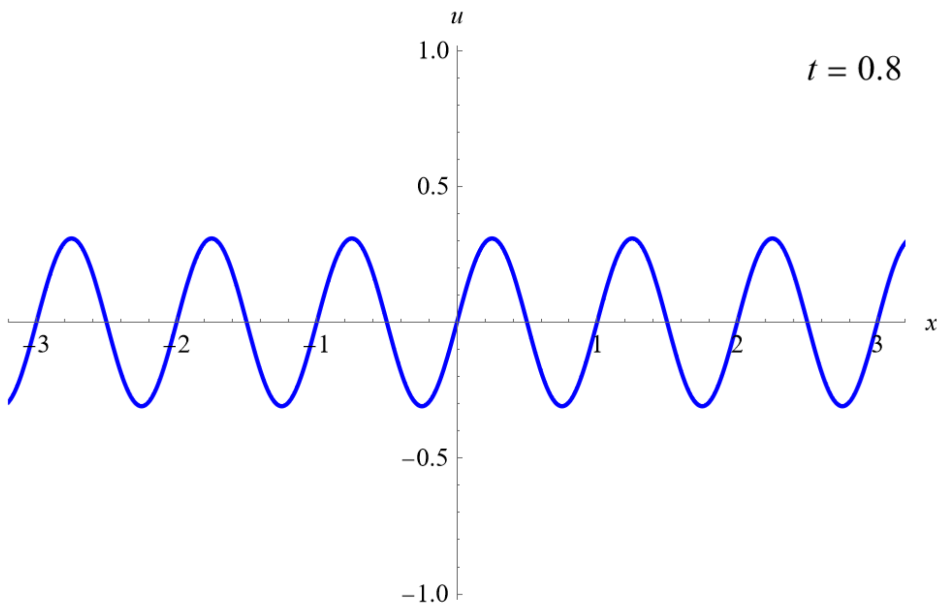

u
1.0
0.5
−0.5
−1.0
−3
−2
−1
1
2
3
x
t = 0.4

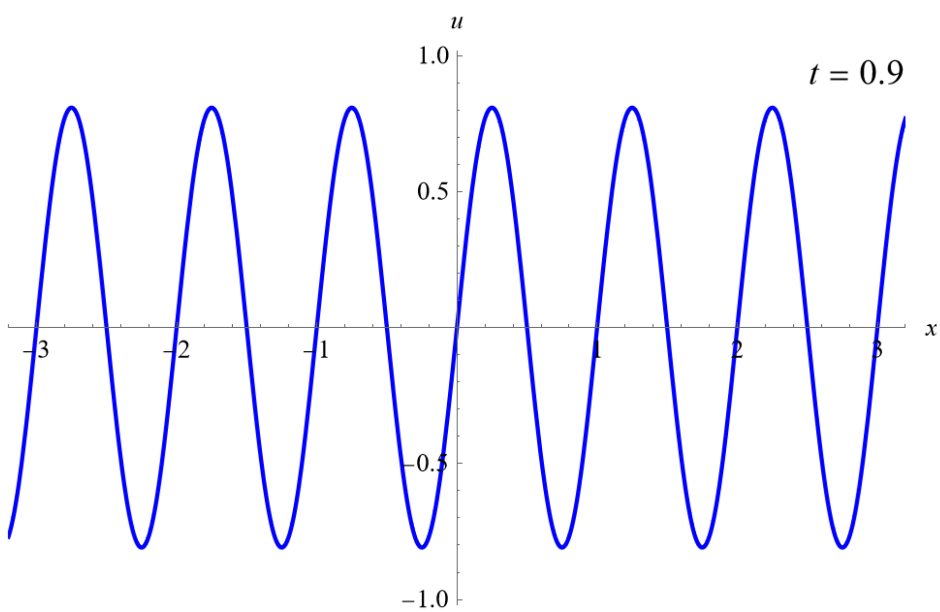

u
1.0
0.5
−0.5
−1.0
x
−3
−2
−1
1
2
3
t = 0.5

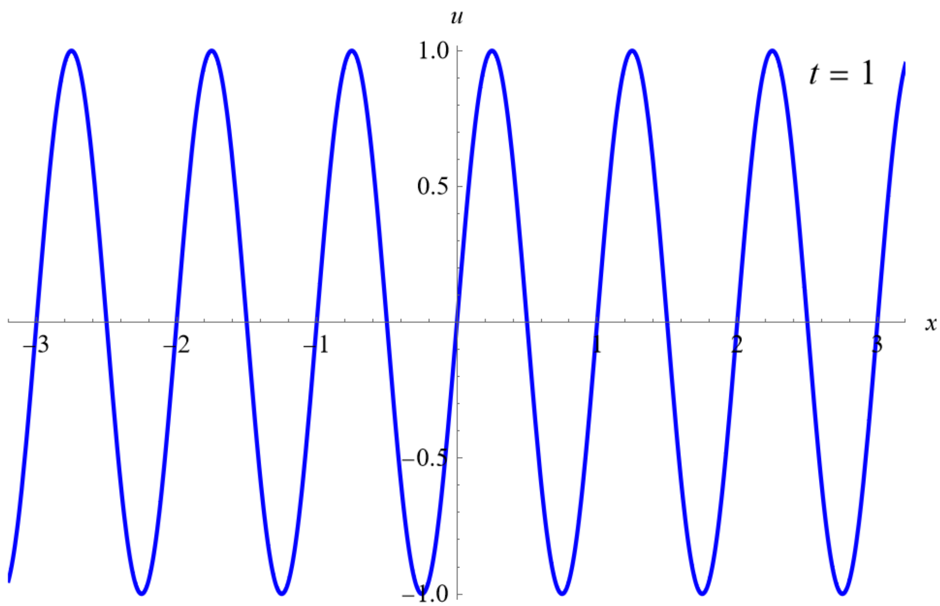

u
1.0
0.5
−0.5
−1.0
x
−3
−2
−1
1
2
3
t = 0.6


u
1.0
0.5
−0.5
−1.0
x
−3
−2
−1
1
2
3
t = 0.7


u
1.0
0.5
−0.5
−1.0
x
−3
−2
−1
1
2
3
t = 0.75


u
1.0
t = 0.8
0.5
x
−3
−2
−1
1
2
3
−0.5
−1.0


u
1.0
t = 0.9
0.5
x
−3
−2
−1
1
2
3
−0.5
−1.0


u
1.0
0.5
−0.5
−1.0
x
−3
−2
−1
1
2
3
t = 1