Exercise 1

The radium in a piece of lead decomposes at a rate which is proportional to the amount present. If 10 percent of the radium decomposes in 200 years, what percent of the original amount of radium will be present in a piece of lead after 1000 years?

Solution

Let the amount of radium in the lead be x. The rate that x decreases with respect to time is proportional to the amount present.

$$\frac{dx}{dt} \propto -x$$

The minus sign is included because the radium decomposes; it doesn't accumulate. Change this proportionality to an equation by introducing a proportionality constant k.

$$\frac{dx}{dt} = -kx$$

Solve this ODE by separating variables.

$$\frac{dx}{x} = -k\,dt$$

Integrate both sides.

$$\int \frac{dx}{x} = -\int k\,dt$$

Evaluate the integrals.

$$\ln|x| = -kt + C$$

The absolute value sign is included since the logarithm argument cannot be negative. Exponentiate both sides.

$$e^{\ln|x|} = e^{-kt+C}$$
$$|x| = e^C e^{-kt}$$

Remove the absolute value sign by placing $\pm$ on the right side.

$$x(t) = \pm e^C e^{-kt}$$

Use a new arbitrary constant A for $\pm e^C$.

$$x(t) = Ae^{-kt}$$

Assuming there's an amount of radium $x = x_0$ initially at $t = 0$, then

$$x(0) = Ae^0 = x_0 \quad \rightarrow \quad A = x_0.$$

As a result, the solution to the ODE is

$$x(t) = x_0 e^{-kt}.$$

Use the fact that 10 percent of the radium decomposes in 200 years to determine k.

$$0.90x_0 = x_0 e^{-k(200)}$$

Solve for k.

$$0.9 = e^{-200k}$$

$$\ln 0.9 = \ln e^{-200k}$$

$$\ln 0.9 = -200k \ln e$$

$$k = -\frac{1}{200} \ln 0.9 \approx 0.000526803$$

The solution to the ODE then becomes

$$x(t) = x_0 \exp\left[\left(\frac{1}{200} \ln 0.9\right) t\right].$$

Therefore, after 1000 years,

$$x(1000) = x_0 \exp\left[\left(\frac{1}{200} \ln 0.9\right) 1000\right] \approx 0.59 x_0 \quad \Rightarrow \quad 59\% \text{ of the radium remains.}$$