Exercise 2

Assume that the half life of the radium in a piece of lead is 1600 years. How much radium will be lost in 100 years?

Solution

Let the amount of radium in the lead be x. The rate that x decreases with respect to time is proportional to the amount present.

$$\frac{dx}{dt} \propto -x$$

The minus sign is included because the radium decomposes; it doesn't accumulate. Change this proportionality to an equation by introducing a proportionality constant k.

$$\frac{dx}{dt} = -kx$$

Solve this ODE by separating variables.

$$\frac{dx}{x} = -k\,dt$$

Integrate both sides.

$$\int \frac{dx}{x} = -\int k\,dt$$

Evaluate the integrals.

$$\ln|x| = -kt + C$$

The absolute value sign is included since the logarithm argument cannot be negative. Exponentiate both sides.

$$\begin{aligned} e^{\ln|x|} &= e^{-kt+C} \\ |x| &= e^C e^{-kt} \end{aligned}$$

Remove the absolute value sign by placing $\pm$ on the right side.

$$x(t) = \pm e^C e^{-kt}$$

Use a new arbitrary constant A for $\pm e^C$.

$$x(t) = Ae^{-kt}$$

Assuming there's an amount of radium $x = x_0$ initially at $t = 0$, then

$$x(0) = Ae^0 = x_0 \quad \rightarrow \quad A = x_0.$$

As a result, the solution to the ODE is

$$x(t) = x_0 e^{-kt}.$$

Use the fact that the half-life of the radium in a piece of lead is 1600 years to determine k.

$$0.50x_0 = x_0 e^{-k(1600)}$$

Solve for k.

$$0.5 = e^{-1600k}$$

$$\ln 0.5 = \ln e^{-1600k}$$

$$\ln 0.5 = -1600k \ln e$$

$$k = -\frac{1}{1600} \ln 0.5 \approx 0.000433217$$

The solution to the ODE then becomes

$$x(t) = x_0 \exp\left[\left(\frac{1}{1600} \ln 0.5\right) t\right].$$

Therefore, after 100 years,

$$x(100) = x_0 \exp\left[\left(\frac{1}{1600} \ln 0.5\right) 100\right] \approx 0.9576 x_0 \quad \Rightarrow \quad 4.24\% \text{ of the radium is lost.}$$