Exercise 3

The following item appeared in a newspaper. "The expedition used the carbon-14 test to measure the amount of radioactivity still present in the organic material found in the ruins, thereby determining that a town existed there as long ago as 7000 B.C." Using the half-life figure of C^{14} as given in the text, determine the approximate percentage of C^{14} still present in the organic material at the time of the discovery.

Solution

Let the amount of carbon-14 in the organic material be x. The rate that x decreases with respect to time is proportional to the amount present.

$$\frac{dx}{dt} \propto -x$$

The minus sign is included because the carbon-14 decomposes; it doesn't accumulate. Change this proportionality to an equation by introducing a proportionality constant k.

$$\frac{dx}{dt} = -kx$$

Solve this ODE by separating variables.

$$\frac{dx}{x} = -k\,dt$$

Integrate both sides.

$$\int \frac{dx}{x} = -\int k\,dt$$

Evaluate the integrals.

$$\ln|x| = -kt + C$$

The absolute value sign is included since the logarithm argument cannot be negative. Exponentiate both sides.

$$e^{\ln|x|} = e^{-kt+C}$$
$$|x| = e^C e^{-kt}$$

Remove the absolute value sign by placing $\pm$ on the right side.

$$x(t) = \pm e^C e^{-kt}$$

Use a new arbitrary constant A for $\pm e^C$.

$$x(t) = Ae^{-kt}$$

Assuming there's an amount of radium $x = x_0$ initially at $t = 0$, then

$$x(0) = Ae^0 = x_0 \quad \rightarrow \quad A = x_0.$$

As a result, the solution to the ODE is

$$x(t) = x_0 e^{-kt}.$$

Use the fact that the half-life of carbon-14 is 5600 years to determine k.

$$0.50x_0 = x_0 e^{-k(5600)}$$

Solve for k.

$$0.5 = e^{-5600k}$$

$$\ln 0.5 = \ln e^{-5600k}$$

$$\ln 0.5 = -5600k \ln e$$

$$k = -\frac{1}{5600} \ln 0.5 \approx 0.0001238$$

The solution to the ODE then becomes

$$x(t) = x_0 \exp\left[\left(\frac{1}{5600} \ln 0.5\right) t\right].$$

The town existed in 7000 B.C., so assuming it was discovered in 2000 A.D., evaluate x at $t = 9000$.

$$x(9000) = x_0 \exp\left[\left(\frac{1}{1600} \ln 0.5\right) 9000\right] \approx 0.3282x_0 \quad \Rightarrow \quad \text{32.82\% of the carbon-14 remains after 9000 years.}$$