Problem 1.11

Determine the Fourier series for the triangular wave shown in Fig. P1.11.

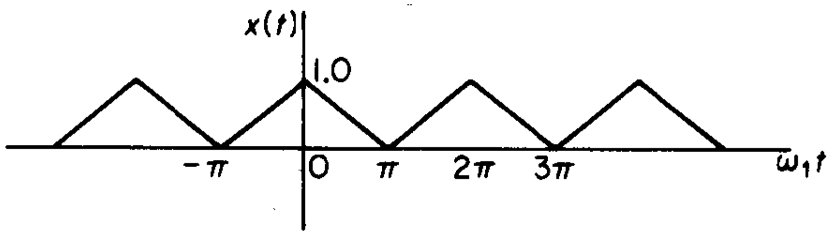

FIGURE P1.11.

Solution

Notice that the wave is even and repeats itself every 2π radians. The general Fourier series for a 2π-periodic function is

$$x(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta. \tag{1}$$

In order to take advantage of the properties of even and odd functions, we aim to find the Fourier series by integrating over the symmetric interval $(-\pi, \pi)$. Integrate both sides of equation (1) with respect to θ from $-\pi$ to π to solve for A_0.

$$\begin{aligned}
\int_{-\pi}^{\pi} x(\theta)\, d\theta &= \int_{-\pi}^{\pi} \left(A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \right) d\theta \\
&= \int_{-\pi}^{\pi} A_0\, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} A_n \cos n\theta\, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} B_n \sin n\theta\, d\theta \\
&= A_0 \int_{-\pi}^{\pi} d\theta + \sum_{n=1}^{\infty} A_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta\, d\theta}_{=0} + \sum_{n=1}^{\infty} B_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta\, d\theta}_{=0} \\
&= 2\pi A_0
\end{aligned}$$

Consequently,

$$\begin{aligned}
A_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\theta)\, d\theta \\
&= \frac{1}{2\pi} \cdot 2 \int_{0}^{\pi} x(\theta)\, d\theta \\
&= \frac{1}{\pi} \int_{0}^{\pi} \left(1 - \frac{1}{\pi}\theta \right) d\theta \\
&= \frac{1}{\pi} \left(\frac{\pi}{2} \right) \\
&= \frac{1}{2}.
\end{aligned}$$

A_n will now be determined. Multiply both sides of equation (1) by $\cos m\theta$, where m is an integer,

$$x(\theta)\cos m\theta = A_0 \cos m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta$$

and then integrate both sides with respect to θ from $-\pi$ to π.

$$\begin{aligned}
\int_{-\pi}^{\pi} x(\theta)\cos m\theta \, d\theta &= \int_{-\pi}^{\pi} \left(A_0 \cos m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta \right) d\theta \\
&= \int_{-\pi}^{\pi} A_0 \cos m\theta \, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta \, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta \, d\theta \\
&= A_0 \underbrace{\int_{-\pi}^{\pi} \cos m\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta}_{= \pi \text{ only if } n = m} + \sum_{n=1}^{\infty} B_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta \cos m\theta \, d\theta}_{= 0 \text{ for all } n \text{ and } m}
\end{aligned}$$

Because the trigonometric functions are orthogonal, only one term in the first infinite series remains as a result of the integration. All other terms vanish.

$$\int_{-\pi}^{\pi} x(\theta)\cos n\theta \, d\theta = A_n(\pi)$$

Consequently,

$$\begin{aligned}
A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x(\theta) \cos n\theta \, d\theta \\
&= \frac{1}{\pi} \cdot 2 \int_0^{\pi} x(\theta) \cos n\theta \, d\theta \\
&= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{1}{\pi}\theta\right) \cos n\theta \, d\theta \\
&= \frac{2}{\pi} \frac{1 - \cos n\pi}{n^2 \pi} \\
&= \frac{2}{\pi} \frac{1 - (-1)^n}{n^2 \pi}.
\end{aligned}$$

B_n will now be determined. Multiply both sides of equation (1) by $\sin m\theta$, where m is an integer,

$$x(\theta)\sin m\theta = A_0 \sin m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta$$

and then integrate both sides with respect to θ from $-\pi$ to π.

$$\begin{aligned}\int_{-\pi}^{\pi} x(\theta)\sin m\theta \, d\theta &= \int_{-\pi}^{\pi} \left(A_0 \sin m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta \right) d\theta \\ &= \int_{-\pi}^{\pi} A_0 \sin m\theta \, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta \, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta \, d\theta \\ &= A_0 \underbrace{\int_{-\pi}^{\pi} \sin m\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta \sin m\theta \, d\theta}_{=0 \text{ for all } n \text{ and } m} + \sum_{n=1}^{\infty} B_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta}_{=\pi \text{ only if } n=m}\end{aligned}$$

Because the trigonometric functions are orthogonal, only one term in the second infinite series remains as a result of the integration. All other terms vanish.

$$\int_{-\pi}^{\pi} x(\theta)\sin n\theta \, d\theta = B_n(\pi)$$

Consequently,

$$\begin{aligned}B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x(\theta)\sin n\theta \, d\theta \\ &= 0\end{aligned}$$

because the integral of an odd function over a symmetric interval is zero. With the coefficients determined, equation (1) becomes

$$x(\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{1-(-1)^n}{n^2\pi} \cos n\theta.$$

If n is even, the coefficient of cosine vanishes, so the series can be simplified (that is, made to converge faster) by summing over the odd integers only. Let $n = 2k-1$ in the sum.

$$x(\theta) = \frac{1}{2} + \sum_{2k-1=1}^{\infty} \frac{2}{\pi} \frac{1-(-1)}{(2k-1)^2\pi} \cos[(2k-1)\theta]$$

Therefore, replacing θ with $\omega_1 t$,

$$x(t) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos[(2k-1)\omega_1 t].$$

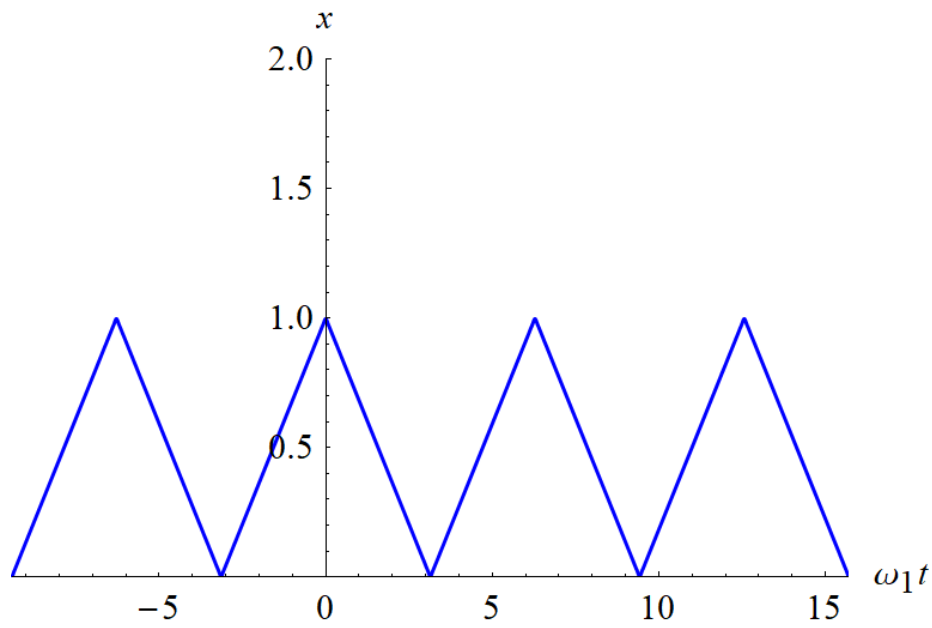

Figure 1: This figure shows a plot of $x(t)$ versus t using only the first 30 terms in the infinite series. The more terms that are used in the series, the more it looks like the function in Figure P1.11.