Problem 1.12

Determine the Fourier series for the sawtooth curve shown in Fig. P1.12. Express the result of Prob. 1.12 in the exponential form of Eq. (1.2.4).

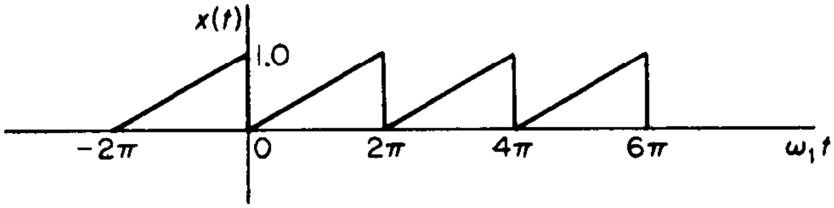

FIGURE P1.12.

Solution

Notice that the wave repeats itself every 2π radians. The general Fourier series for a 2π-periodic function is

$$x(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta. \tag{1}$$

Integrate both sides of equation (1) with respect to θ from 0 to 2π to solve for A_0.

$$\begin{aligned}
\int_0^{2\pi} x(\theta)\,d\theta &= \int_0^{2\pi} \left(A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \right) d\theta \\
&= \int_0^{2\pi} A_0\,d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} A_n \cos n\theta\,d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} B_n \sin n\theta\,d\theta \\
&= A_0 \int_0^{2\pi} d\theta + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{2\pi} \cos n\theta\,d\theta}_{=0} + \sum_{n=1}^{\infty} B_n \underbrace{\int_0^{2\pi} \sin n\theta\,d\theta}_{=0} \\
&= 2\pi A_0
\end{aligned}$$

Consequently,

$$\begin{aligned}
A_0 &= \frac{1}{2\pi} \int_0^{2\pi} x(\theta)\,d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2\pi}\theta \right) d\theta \\
&= \frac{1}{4\pi^2} \int_0^{2\pi} \theta\,d\theta \\
&= \frac{1}{4\pi^2} \left(\frac{4\pi^2}{2} \right) \\
&= \frac{1}{2}.
\end{aligned}$$

A_n will now be determined. Multiply both sides of equation (1) by $\cos m\theta$, where m is an integer,

$$x(\theta)\cos m\theta = A_0 \cos m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta$$

and then integrate both sides with respect to θ from 0 to 2π.

$$\begin{aligned}
\int_0^{2\pi} x(\theta)\cos m\theta\, d\theta &= \int_0^{2\pi} \left(A_0 \cos m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta \right) d\theta \\
&= \int_0^{2\pi} A_0 \cos m\theta\, d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta\, d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta\, d\theta \\
&= A_0 \underbrace{\int_0^{2\pi} \cos m\theta\, d\theta}_{=0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{2\pi} \cos n\theta \cos m\theta\, d\theta}_{=\pi \text{ only if } n=m} + \sum_{n=1}^{\infty} B_n \underbrace{\int_0^{2\pi} \sin n\theta \cos m\theta\, d\theta}_{=0 \text{ for all } n \text{ and } m}
\end{aligned}$$

Because the trigonometric functions are orthogonal, only one term in the first infinite series remains as a result of the integration. All other terms vanish.

$$\int_0^{2\pi} x(\theta)\cos n\theta\, d\theta = A_n(\pi)$$

Consequently,

$$\begin{aligned}
A_n &= \frac{1}{\pi}\int_0^{2\pi} x(\theta)\cos n\theta\, d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{1}{2\pi}\theta\right)\cos n\theta\, d\theta \\
&= \frac{1}{2\pi^2}\int_0^{2\pi} \theta \cos n\theta\, d\theta \\
&= \frac{1}{2\pi^2}(0) \\
&= 0.
\end{aligned}$$

B_n will now be determined. Multiply both sides of equation (1) by $\sin m\theta$, where m is an integer,

$$x(\theta)\sin m\theta = A_0 \sin m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta$$

and then integrate both sides with respect to θ from 0 to 2π.

$$\begin{aligned}
\int_0^{2\pi} x(\theta)\sin m\theta\, d\theta &= \int_0^{2\pi} \left(A_0 \sin m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta \right) d\theta \\
&= \int_0^{2\pi} A_0 \sin m\theta\, d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta\, d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta\, d\theta \\
&= A_0 \underbrace{\int_0^{2\pi} \sin m\theta\, d\theta}_{=0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{2\pi} \cos n\theta \sin m\theta\, d\theta}_{=0 \text{ for all } n \text{ and } m} + \sum_{n=1}^{\infty} B_n \underbrace{\int_0^{2\pi} \sin n\theta \sin m\theta\, d\theta}_{=\pi \text{ only if } n=m}
\end{aligned}$$

Because the trigonometric functions are orthogonal, only one term in the second infinite series remains as a result of the integration. All other terms vanish.

$$\int_0^{2\pi} x(\theta)\sin n\theta\, d\theta = B_n(\pi)$$

Consequently,

$$\begin{aligned}
B_n &= \frac{1}{\pi}\int_0^{2\pi} x(\theta)\sin n\theta\, d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{1}{2\pi}\theta\right)\sin n\theta\, d\theta \\
&= \frac{1}{2\pi^2}\int_0^{2\pi} \theta \sin n\theta\, d\theta \\
&= \frac{1}{2\pi^2}\left(-\frac{2\pi}{n}\right) \\
&= -\frac{1}{n\pi}.
\end{aligned}$$

The Fourier series for the sawtooth function is then

$$\begin{aligned}
x(\theta) &= A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \\
&= \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{1}{n\pi}\right)\sin n\theta \\
&= \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}.
\end{aligned}$$

Therefore, replacing θ with $\omega_1 t$,

$$x(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\omega_1 t}{n}.$$

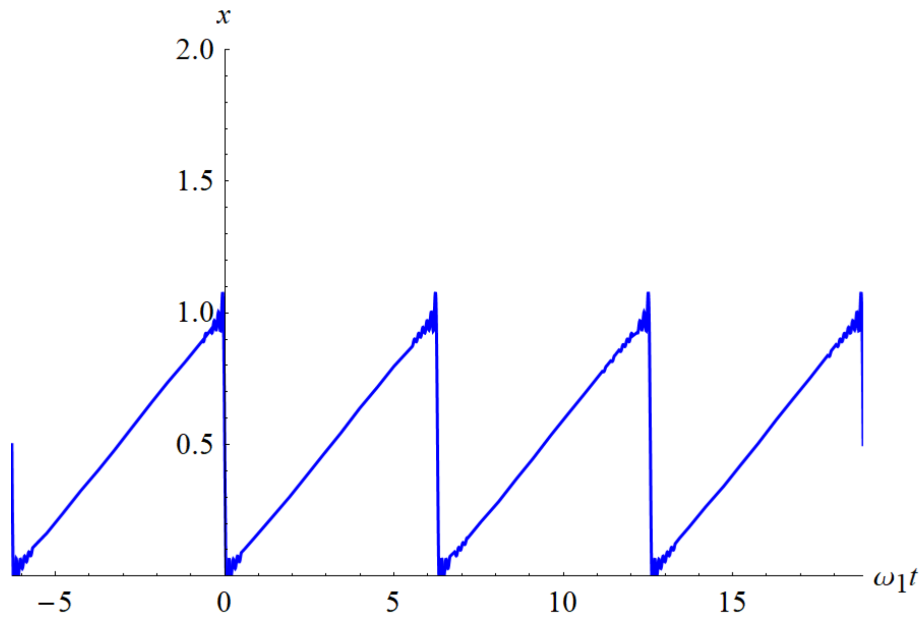

Figure 1: This figure shows a plot of $x(t)$ versus $\omega_1 t$ using the first 50 terms of the infinite series.

In order to obtain the complex Fourier series, write sine in terms of complex exponentials.

$$\begin{aligned} x(t) &= \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{e^{in\omega_1 t} - e^{-in\omega_1 t}}{2i}\right) \\ &= \frac{1}{2} - \sum_{n=1}^{\infty}\left(\frac{e^{in\omega_1 t}}{2\pi in} - \frac{e^{-in\omega_1 t}}{2\pi in}\right) \\ &= \frac{1}{2} - \sum_{n=1}^{\infty}\frac{e^{in\omega_1 t}}{2\pi in} + \sum_{n=1}^{\infty}\frac{e^{-in\omega_1 t}}{2\pi in} \end{aligned}$$

Substitute $n = k$ in the first infinite series and $n = -k$ in the second infinite series.

$$\begin{aligned} &= \frac{1}{2} - \sum_{k=1}^{\infty}\frac{e^{ik\omega_1 t}}{2\pi ik} + \sum_{-k=1}^{\infty}\frac{e^{-i(-k)\omega_1 t}}{2\pi i(-k)} \\ &= \frac{1}{2} - \sum_{k=1}^{\infty}\frac{e^{ik\omega_1 t}}{2\pi ik} - \sum_{k=-\infty}^{-1}\frac{e^{ik\omega_1 t}}{2\pi ik} \\ &= \frac{1}{2} + \sum_{k=1}^{\infty}\frac{i}{2\pi k}e^{ik\omega_1 t} + \sum_{k=-\infty}^{-1}\frac{i}{2\pi k}e^{ik\omega_1 t} \end{aligned}$$

Therefore, the complex Fourier series of $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega_1 t},$$

where

$$c_0 = \frac{1}{2} \quad \text{and} \quad c_k = \frac{i}{2\pi k}, \quad k \neq 0.$$