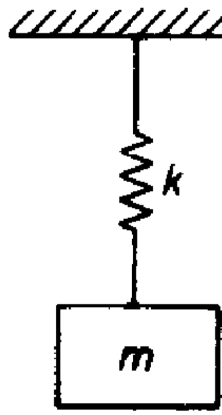
Problem 2.3

A 4.53-kg mass attached to the lower end of a spring whose upper end is fixed vibrates with a natural period of 0.45 s. Determine the natural period when a 2.26-kg mass is attached to the midpoint of the same spring with the upper and lower ends fixed.

Solution



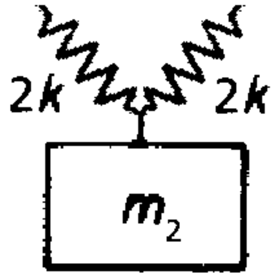
The spring stiffness k can be determined from the information in the first sentence. Use the formula for the natural frequency f and the fact that the period is $\tau = 1/f$.

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$
$$\frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$
$$\frac{2\pi}{\tau} = \sqrt{\frac{k}{m}}$$
$$\frac{4\pi^2}{\tau^2} = \frac{k}{m}$$
$$k = \frac{4m\pi^2}{\tau^2}$$

Plugging in the numbers, we have

$$k = \frac{4(4.53 \text{ kg})\pi^2}{(0.45 \text{ s})^2} \approx 883 \frac{\text{kg}}{\text{s}^2}.$$

Now that k is known, we can figure out what the period will be for a different mass m_2 hanging from the middle of the spring. Because the mass is hanging from the midpoint, the spring is effectively split into two, each with half the length and hence with spring stiffness $2k$.



These two springs are in parallel can be replaced with one effective spring stiffness,

$$k_{\text{eff}} = 2k + 2k = 4k.$$

Obtain the period from the formula for the natural frequency.

$$\begin{aligned} f_2 &= \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{m_2}} \\ \frac{1}{\tau_2} &= \frac{1}{2\pi}\sqrt{\frac{4k}{m_2}} \\ \frac{1}{\tau_2} &= \frac{1}{\pi}\sqrt{\frac{k}{m_2}} \\ \tau_2 &= \pi\sqrt{\frac{m_2}{k}} \end{aligned}$$

Therefore, the natural period for the 2.26-kg mass is

$$\tau_2 = \pi\sqrt{\frac{2.26 \text{ kg}}{883 \ \frac{\text{kg}}{\text{s}^2}}} \approx 0.195 \text{ s}.$$