Exercise 12

Find the particular solution for each of the following initial value problems:

$$u' + (\tan x)u = \cos x, \quad u(0) = 1$$

Solution

This is an inhomogeneous first order linear ODE, so we can multiply both sides by the integrating factor,

$$I(x) = e^{\int \tan x \, dx} = e^{-\ln \cos x} = \sec x,$$

to solve it. The equation becomes

$$(\sec x)u' + (\sec x)(\tan x)u = 1.$$

Observe that the left side can be written as $[(\sec x)u]'$ by the product rule.

$$\frac{d}{dx}[(\sec x)u] = 1$$

Now integrate both sides with respect to x.

$$(\sec x)u = x + C$$

The general solution is thus

$$u(x) = \cos x(x + C).$$

Because an initial condition is given, this constant of integration can be determined.

$$u(0) = \cos 0(0 + C) = C \quad \rightarrow \quad C = 1$$

Therefore,

$$u(x) = \cos x(x + 1).$$