Exercise 1

Find the general solution for the following second order ODEs:

$$u'' - 4u' + 4u = 0$$

Solution

This is a homogeneous linear ODE with constant coefficients, so the solution is of the form, $u = e^{rx}$.

$$u = e^{rx} \quad \to \quad u' = re^{rx} \quad \to \quad u'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} - 4re^{rx} + 4e^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 - 4r + 4 = 0$$

Factor the left side.

$$(r-2)^2 = 0$$

$r = 2$ with a multiplicity of 2. Therefore, the general solution is

$$u(x) = C_1 e^{2x} + C_2 x e^{2x}.$$

We can check that this is the solution. The first and second derivatives are

$$u' = e^{2x}(2C_1 + C_2 + 2C_2 x)$$
$$u'' = 4e^{2x}(C_1 + C_2 + C_2 x).$$

Hence,

$$u'' - 4u' + 4u = 4e^{2x}(\cancel{C_1} + \cancel{C_2} + \cancel{C_2 x}) - 4e^{2x}(\cancel{2C_1} + \cancel{C_2} + \cancel{2C_2 x}) + 4e^{2x}(\cancel{C_1} + \cancel{C_2 x}) = 0,$$

which means this is the correct solution.