Exercise 12

Find the general solution for the following initial value problems:

$$u'' - 9u = 0, \quad u(0) = 1,\ u'(0) = 0$$

Solution

This is a homogeneous linear ODE with constant coefficients, so the solution is of the form, $u = e^{rx}$.

$$u = e^{rx} \quad \rightarrow \quad u' = re^{rx} \quad \rightarrow \quad u'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} - 9e^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 - 9 = 0$$

Factor the left side.

$$(r + 3)(r - 3) = 0$$

$r = -3$ or $r = 3$, so the general solution is

$$u(x) = C_1 e^{-3x} + C_2 e^{3x}.$$

Because we have two initial conditions, we can determine C_1 and C_2.

$$u'(x) = -3C_1 e^{-3x} + 3C_2 e^{3x}$$

$$\begin{aligned} u(0) &= C_1 + C_2 = 1 \\ u'(0) &= -3C_1 + 3C_2 = 0 \end{aligned}$$

Solving this system of equations gives $C_1 = 1/2$ and $C_2 = 1/2$. Therefore,

$$u(x) = \frac{1}{2}e^{-3x} + \frac{1}{2}e^{3x} = \cosh 3x.$$

We can check that this is the solution. The first and second derivatives are

$$\begin{aligned} u' &= 3\sinh 3x \\ u'' &= 9\cosh 3x. \end{aligned}$$

Hence,

$$u'' - 9u = \cancel{9\cosh 3x} - \cancel{9\cosh 3x} = 0,$$

which means this is the correct solution.