Exercise 13

Use the method of undetermined coefficients to find the general solution for the following second order ODEs:

$$u'' - u' = 1$$

Solution

This is an inhomogeneous ODE, so the general solution is the sum of the complementary and particular solutions.

$$u = u_c + u_p$$

The complementary solution is the solution to the associated homogeneous equation,

$$u_c'' - u_c' = 0.$$

This is a linear ODE with constant coefficients, so the solution will be of the form $u_c = e^{rx}$.

$$u_c = e^{rx} \quad \rightarrow \quad u_c' = re^{rx} \quad \rightarrow \quad u_c'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} - re^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 - r = 0$$

Factor the left side.

$$r(r-1) = 0$$

$r = 0$ or $r = 1$, so the complementary solution is

$$u_c(x) = C_1 + C_2 e^x.$$

Now we turn our attention to the particular solution. Because the inhomogeneous term, 1, is a constant, the particular solution should be chosen so that the higher derivatives vanish but the smallest derivative remains, i.e. $u_p = Ax$. Plugging this form into the ODE yields $-A = 1$, which means $A = -1$. Thus, $u_p = -x$. Therefore, the general solution to the ODE is

$$u(x) = C_1 + C_2 e^x - x.$$

We can check that this is the solution. The first and second derivatives are

$$u' = C_2 e^x - 1$$
$$u'' = C_2 e^x.$$

Hence,

$$u'' - u' = \cancel{C_2 e^x} - (\cancel{C_2 e^x} - 1) = 1,$$

which means this is the correct solution.