Exercise 16

Use the method of undetermined coefficients to find the general solution for the following second order ODEs:

$$u'' - u = 2\cos x$$

Solution

This is an inhomogeneous ODE, so the general solution is the sum of the complementary and particular solutions.

$$u = u_c + u_p$$

The complementary solution is the solution to the associated homogeneous equation,

$$u_c'' - u_c = 0.$$

This is a linear ODE with constant coefficients, so the solution will be of the form $u_c = e^{rx}$.

$$u_c = e^{rx} \quad \rightarrow \quad u_c' = re^{rx} \quad \rightarrow \quad u_c'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} - e^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 - 1 = 0$$

Factor the left side.

$$(r+1)(r-1) = 0$$

$r = -1$ or $r = 1$, so the complementary solution is

$$u_c(x) = C_1 e^{-x} + C_2 e^x.$$

We can write this in terms of hyperbolic sine and hyperbolic cosine.

$$u_c(x) = A\cosh x + B\sinh x$$

Now we turn our attention to the particular solution. Because the inhomogeneous term is $2\cos x$ and u' is not present on the left side, try a particular solution of the form, $u_p = C\cos x$. Plugging this form into the ODE yields $-C\cos x - C\cos x = 2\cos x$, which means $C = -1$. Thus, $u_p = -\cos x$. Therefore, the general solution to the ODE is

$$u(x) = A\cosh x + B\sinh x - \cos x.$$

We can check that this is the solution. The first and second derivatives are

$$\begin{aligned} u' &= A\sinh x + B\cosh x + \sin x \\ u'' &= A\cosh x + B\sinh x + \cos x. \end{aligned}$$

Hence,

$$u'' - u = \cancel{A\cosh x} + \cancel{B\sinh x} + \cos x - (\cancel{A\cosh x} + \cancel{B\sinh x} - \cos x) = 2\cos x,$$

which means this is the correct solution.

This answer is in disagreement with the answer at the back of the book, $u(x) = A\cosh x - \cos x$. Since the ODE is of second order, there have to be two constants in the general solution.