Exercise 18

Use the method of undetermined coefficients to find the particular solution for the following initial value problems:

$$u'' + u = 6e^x, \quad u(0) = 3,\ u'(0) = 2$$

Solution

This is an inhomogeneous ODE, so the general solution is the sum of the complementary and particular solutions.

$$u = u_c + u_p$$

The complementary solution is the solution to the associated homogeneous equation,

$$u_c'' + u_c = 0.$$

This is a linear ODE with constant coefficients, so the solution will be of the form $u_c = e^{rx}$.

$$u_c = e^{rx} \quad \rightarrow \quad u_c' = re^{rx} \quad \rightarrow \quad u_c'' = r^2e^{rx}$$

Substituting these into the equation gives us

$$r^2e^{rx} + e^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 + 1 = 0$$

Factor the left side.

$$(r + i)(r - i) = 0$$

$r = -i$ or $r = i$, so the complementary solution is

$$u_c(x) = C_1e^{-ix} + C_2e^{ix}.$$

We can write this in terms of sine and cosine by Euler's formula.

$$u_c(x) = A\cos x + B\sin x$$

Now we turn our attention to the particular solution. Because the inhomogeneous term is $6e^x$, try the particular solution, $u_p = Ce^x$. Plugging this form into the ODE yields $Ce^x + Ce^x = 6e^x$, which means $C = 3$. Thus, $u_p = 3e^x$. Therefore, the general solution to the ODE is

$$u(x) = A\cos x + B\sin x + 3e^x.$$

These constants can be determined since initial conditions are given.

$$u'(x) = -A\sin x + B\cos x + 3e^x$$

$$u(0) = A + 3 = 3$$
$$u'(0) = B + 3 = 2$$

The solution to this system of equations is $A = 0$ and $B = -1$. Therefore,

$$u(x) = 3e^x - \sin x.$$

We can check that this is the solution. The first and second derivatives are

$$u' = 3e^x - \cos x$$
$$u'' = 3e^x + \sin x.$$

Hence,

$$u'' + u = 3e^x + \cancel{\sin x} + 3e^x - \cancel{\sin x} = 6e^x,$$

which means this is the correct solution.