Exercise 8

Find the general solution for the following initial value problems:

$$u'' - 6u' + 9u = 0, \quad u(0) = 1,\ u'(0) = 4$$

Solution

This is a homogeneous linear ODE with constant coefficients, so the solution is of the form, $u = e^{rx}$.

$$u = e^{rx} \quad \to \quad u' = re^{rx} \quad \to \quad u'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} - 6re^{rx} + 9e^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 - 6r + 9 = 0$$

Factor the left side.

$$(r-3)^2 = 0$$

$r = 3$ with a multiplicity of 2. Therefore, the general solution is

$$u(x) = C_1 e^{3x} + C_2 x e^{3x}.$$

Because we have two initial conditions, we can determine C_1 and C_2.

$$u'(x) = e^{3x}(3C_1 + C_2 + 3C_2 x)$$

$$\begin{aligned} u(0) &= C_1 e^0 = 1 && \to \quad C_1 = 1 \\ u'(0) &= e^0(3C_1 + C_2) = 4 && \to \quad C_2 = 1 \end{aligned}$$

Therefore,

$$u(x) = e^{3x} + xe^{3x}.$$

We can check that this is the solution. The first and second derivatives are

$$\begin{aligned} u' &= e^{3x}(4 + 3x) \\ u'' &= 3e^{3x}(5 + 3x). \end{aligned}$$

Hence,

$$u'' - 6u' + 9u = 3e^{3x}(\cancel{5} + \cancel{3x}) - 6e^{3x}(\cancel{4} + \cancel{3x}) + 9e^{3x}(\cancel{1} + \cancel{x}) = 0,$$

which means this is the correct solution.