Exercise 2

Find the series solution for the following homogeneous second order ODEs:

$$u'' - xu' + xu = 0$$

Solution

Because $x = 0$ is an ordinary point, the series solution of this differential equation will be of the form,

$$u(x) = \sum_{n=0}^{\infty} a_n x^n.$$

To determine the coefficients, a_n, we will have to plug the form into the ODE. Before we can do so, though, we must write expressions for u' and u''.

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad u'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad u''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Now we substitute these series into the ODE.

$$u'' - xu' + xu = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

The first series on the left is zero for $n = 0$ and $n = 1$, so we can start the sum from $n = 2$. In addition, the second series is zero for $n = 0$, so we can start the sum from $n = 1$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Since we want to combine the series, we want the first two series to start from $n = 0$. We can start the first at $n = 0$ as long as we replace n with $n + 2$, and we can start the second at $n = 0$ as long as we replace n with $n + 1$.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n} - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

To get x^{n+1} in the first series, write out the first term and change n to $n + 1$.

$$2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Now that we have x^{n+1} in every series, we can combine the series.

$$2a_2 + \sum_{n=0}^{\infty} [(n+3)(n+2) a_{n+3} x^{n+1} - (n+1) a_{n+1} x^{n+1} + a_n x^{n+1}] = 0$$

Factor the left side.

$$2a_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)a_{n+3} - (n+1)a_{n+1} + a_n]x^{n+1} = 0$$

Thus,

$$\begin{aligned} 2a_2 = 0 \quad &\text{and} \quad (n+3)(n+2)a_{n+3} - (n+1)a_{n+1} + a_n = 0 \\ a_2 = 0 \quad &\text{and} \quad a_{n+3} = \frac{(n+1)a_{n+1} - a_n}{(n+3)(n+2)}. \end{aligned}$$

Now that we know the recurrence relation, we can determine the coefficients.

$$\begin{aligned} n = 0: \qquad & a_3 = \frac{a_1 - a_0}{6} \\ n = 1: \qquad & a_4 = \frac{2a_2 - a_1}{12} = -\frac{a_1}{12} \\ n = 2: \qquad & a_5 = \frac{3a_3 - a_2}{20} = \frac{3a_3}{20} = \frac{1}{40}(a_1 - a_0) \\ n = 3: \qquad & a_6 = \frac{4a_4 - a_3}{30} = \frac{1}{30}\left[-\frac{1}{6}(a_1 - a_0) + 4\left(-\frac{a_0}{12}\right)\right] = \frac{1}{180}(a_0 - 3a_1) \\ n = 4: \qquad & a_7 = \frac{5a_5 - a_4}{42} = \frac{1}{42}\left[-\left(-\frac{a_1}{12}\right) + 5 \cdot \frac{1}{40}(a_1 - a_0)\right] = \frac{1}{1008}(-3a_0 + 5a_1) \\ & \vdots \qquad \vdots \end{aligned}$$

Therefore,

$$\begin{aligned} u(x) = a_0 \left(1 - \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{180}x^6 - \frac{1}{336}x^7 + \cdots\right) \\ + a_1 \left(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{40}x^5 - \frac{1}{60}x^6 + \frac{5}{1008}x^7 + \cdots\right), \end{aligned}$$

where a_0 and a_1 are arbitrary constants.