Exercise 5

Find the series solution for the following inhomogeneous second order ODEs:

$$u'' - u' + xu = \sin x$$

Solution

Because $x = 0$ is an ordinary point, the series solution of this differential equation will be of the form,

$$u(x) = \sum_{n=0}^{\infty} a_n x^n.$$

To determine the coefficients, a_n, we will have to plug the form into the ODE. Before we can do so, though, we must write expressions for u' and u''.

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad u'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad u''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Also, the Taylor series of $\sin x$ about $x = 0$ is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Now we substitute these series into the ODE.

$$u'' - u' + xu = \sin x$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

The first series on the left is zero for $n = 0$ and $n = 1$, so we can start the sum from $n = 2$. In addition, the second series is zero for $n = 0$, so we can start the sum from $n = 1$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Since we want to combine the series on the left, we want the first two series to start from $n = 0$. We can start the first at $n = 0$ as long as we replace n with $n + 2$, and we can start the second at $n = 0$ as long as we replace n with $n + 1$.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

To get x^{n+1} in the first two series, write out the first term and change n to $n + 1$ in each.

$$2a_2 - a_1 + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

The point of doing this is so that x^{n+1} is present in each term so we can combine the series.

$$2a_2 - a_1 + \sum_{n=0}^{\infty}[(n+3)(n+2)a_{n+3}x^{n+1} - (n+2)a_{n+2}x^{n+1} + a_n x^{n+1}] = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Factor the left side.

$$2a_2 - a_1 + \sum_{n=0}^{\infty}[(n+3)(n+2)a_{n+3} - (n+2)a_{n+2} + a_n]x^{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

We can split the series on the left into two: one for when n is even ($n = 2k$) and another for when n is odd ($n = 2k+1$).

$$\begin{aligned} 2a_2 - a_1 &+ \sum_{k=0}^{\infty}[(2k+3)(2k+2)a_{2k+3} - (2k+2)a_{2k+2} + a_{2k}]x^{2k+1} \\ &+ \sum_{k=0}^{\infty}[(2k+4)(2k+3)a_{2k+4} - (2k+3)a_{2k+3} + a_{2k+1}]x^{2k+2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \end{aligned}$$

Note that k and n are just dummy indices, so we can put $n = k$ on the right side. Now we match coefficients on both sides.

$$\begin{aligned} &2a_2 - a_1 = 0 \\ &(2k+3)(2k+2)a_{2k+3} - (2k+2)a_{2k+2} + a_{2k} = \frac{(-1)^k}{(2k+1)!} \\ &(2k+4)(2k+3)a_{2k+4} - (2k+3)a_{2k+3} + a_{2k+1} = 0 \end{aligned}$$

Now that we know the recurrence relations, we can determine a_n.

$$\begin{aligned} & & 2a_2 - a_1 = 0 \quad &\rightarrow \quad a_2 = \frac{1}{2}a_1 \\ n = 0: & & 6a_3 - 2a_2 + a_0 = 1 \quad &\rightarrow \quad a_3 = \frac{1}{6}(1 - a_0 + a_1) \\ n = 1: & & 12a_4 - 3a_3 + a_1 = 0 \quad &\rightarrow \quad a_4 = \frac{1}{24}(1 - a_0 - a_1) \\ n = 2: & & 20a_5 - 4a_4 + a_2 = -\frac{1}{6} \quad &\rightarrow \quad a_5 = \frac{1}{120}(-a_0 - 4a_1) \\ n = 3: & & 30a_6 - 5a_5 + a_3 = 0 \quad &\rightarrow \quad a_6 = \frac{1}{720}(-4 + 3a_0 - 8a_1) \\ & & \vdots \quad & \quad \vdots \end{aligned}$$

Therefore,

$$\begin{aligned} u(x) = a_0 &\left(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{240}x^6 + \frac{1}{630}x^7 + \cdots\right) \\ &+ a_1\left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{30}x^5 - \frac{1}{90}x^6 - \frac{1}{1680}x^7 + \cdots\right) \\ &+ \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{180}x^6 - \frac{1}{630}x^7 + \cdots, \end{aligned}$$

where a_0 and a_1 are arbitrary constants.