Exercise 14

Differentiate the following $F(x)$ as many times as you need to get rid of the integral sign:

$$F(x) = x^2 + \int_0^x (x-t)^2 u(t)\,dt$$

Solution

Take the derivative of both sides with respect to x and use the Leibnitz rule on the integral.

$$F'(x) = 2x + 0 \cdot 1 - x^2 u(0) \cdot 0 + \int_0^x \frac{\partial}{\partial x}(x-t)^2 u(t)\,dt$$

The first derivative of $F(x)$ is thus

$$F'(x) = 2x + \int_0^x 2(x-t)u(t)\,dt.$$

Differentiate both sides once more with respect to x, again using the Leibnitz rule.

$$F''(x) = 2 + 2\left[0 \cdot 1 - xu(0) \cdot 0 + \int_0^x \frac{\partial}{\partial x}(x-t)u(t)\,dt\right]$$

The second derivative of $F(x)$ is thus

$$F''(x) = 2 + 2\int_0^x u(t)\,dt.$$

Differentiate both sides once more with respect to x.

$$F'''(x) = 0 + 2\frac{d}{dx}\int_0^x u(t)\,dt$$

The fundamental theorem of calculus can be applied here to eliminate the integral sign. The third derivative of $F(x)$ is thus

$$F'''(x) = 2u(x).$$