Exercise 17

Use Leibnitz rule to prove the following identities:

$$F(x) = \int_0^x (x-t)^n u(t)\,dt, \text{ show that } F^{(n+1)} = n!u(x), \quad n \geq 0.$$

Solution

We will prove this identity by mathematical induction. There are three steps involved in the procedure.

1. Check the base case $n = 0$.
2. Assume the inductive hypothesis.
3. Show the result is true for $n = k + 1$.

Step 1: Check the Base Case $n = 0$

Setting $n = 0$ yields

$$F(x) = \int_0^x u(t)\,dt.$$

If we differentiate both sides with respect to x, it gives us

$$F'(x) = \frac{d}{dx}\int_0^x u(t)\,dt.$$

According to the fundamental theorem of calculus, we have

$$F'(x) = u(x)$$

for the first derivative. Plugging in $n = 0$ into the result gives us the same thing.

$$F^{(0+1)} = 0!u(x)$$

That is,

$$F'(x) = u(x).$$

Hence, the result is true for $n = 0$, the base case.

Step 2: Assume the Inductive Hypothesis

Now we assume the inductive hypothesis, that is, that the result is true for $n = k$.

$$F(x) = \int_0^x (x-t)^k u(t)\,dt$$

implies that

$$F^{(k+1)} = k!u(x).$$

Step 3: Show the Result is True for $n = k + 1$

Our task in the final step is to show that

$$F(x) = \int_0^x (x-t)^{k+1} u(t)\, dt$$

implies that

$$F^{(k+2)} = (k+1)! u(x).$$

Start by differentiating $F(x)$ with respect to x, using the Leibniz rule on the integral.

$$F'(x) = 0 \cdot 1 - x^{k+1} u(0) \cdot 0 + \int_0^x \frac{\partial}{\partial x} (x-t)^{k+1} u(t)\, dt$$

The first derivative is thus

$$F'(x) = (k+1) \int_0^x (x-t)^k u(t)\, dt.$$

The inductive hypothesis tells us that if we take $k+1$ derivatives of the integral above, it will be equal to $k! u(x)$, so let's do that. Take $k+1$ derivatives with respect to x on both sides.

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}} F'(x) &= (k+1) \frac{d^{k+1}}{dx^{k+1}} \int_0^x (x-t)^k u(t)\, dt \\ F^{k+2} &= (k+1) k! u(x) \\ F^{k+2} &= (k+1)! u(x). \end{aligned}$$

Therefore, by mathematical induction, if

$$F(x) = \int_0^x (x-t)^n u(t)\, dt,$$

then

$$F^{(n+1)} = n! u(x), \quad n \geq 0.$$