Exercise 18

Use Leibnitz rule to prove the following identities:

$$F(x) = \int_0^x t^n (x-t)^m \, dt, \text{ show that } F^{(m)} = \frac{m!}{n+1} x^{n+1},$$

m and n are positive integers.

Solution

Observe that $F(x)$ has the same form as the previous exercise, so we can use the result we proved there. Specifically, if

$$F(x) = \int_0^x (x-t)^n u(t) \, dt,$$

then

$$F^{(n+1)} = n! u(x), \quad n \geq 0.$$

In this exercise, $u(t) = t^n$ and the exponent of $(x-t)$ is m. Thus,

$$F^{(m+1)} = m! x^n.$$

Now we can simply integrate both sides to get the desired result.

$$\frac{d}{dx} F^{(m)} = m! x^n$$

$$F^{(m)} = \int m! x^n \, dx$$

Therefore,

$$F^{(m)} = \frac{m!}{n+1} x^{n+1},$$

where m and n are positive integers.