Exercise 3

Prove the following:

$$\int_0^x \int_0^{x_1} (x-t)u(x_1)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x-t)^2 u(x_1)\,dt\,dx_1 = \frac{1}{6}\int_0^x (x-t)^2(3+2(x-t))u(t)\,dt$$

[TYPO: The integrands should be $(x_1 - t)u(t)$ and $(x_1 - t)^2 u(t)$, respectively.]

Solution

From Right to Left

Let

$$G(x) = \frac{1}{6}\int_0^x (x-t)^2[3+2(x-t)]u(t)\,dt.$$

Note that $G(0) = 0$. Differentiate both sides with respect to x and use the Leibniz integration rule.

$$\begin{aligned}
G'(x) &= \frac{1}{6}\frac{d}{dx}\int_0^x (x-t)^2[3+2(x-t)]u(t)\,dt \\
&= \frac{1}{6}\int_0^x \frac{\partial}{\partial x}(x-t)^2[3+2(x-t)]u(t)\,dt + \frac{1}{6}(0)^2[3+2(0)]u(x)\cdot 1 - \frac{1}{6}x^2[3+2(x)]u(0)\cdot 0 \\
&= \frac{1}{6}\int_0^x \{2(x-t)[3+2(x-t)] + (x-t)^2(2)\}u(t)\,dt \\
&= \frac{1}{6}\int_0^x [6(x-t)+6(x-t)^2]u(t)\,dt \\
&= \int_0^x [(x-t)+(x-t)^2]u(t)\,dt
\end{aligned}$$

Now integrate both sides with respect to x.

$$G(x) = \int^x \int_0^{x_1} [(x_1-t)+(x_1-t)^2]u(t)\,dt\,dx_1 + C$$

Set the constant of integration and the lower limit of integration to 0 in order to satisfy $G(0) = 0$.

$$\begin{aligned}
G(x) &= \int_0^x \int_0^{x_1} [(x_1-t)+(x_1-t)^2]u(t)\,dt\,dx_1 \\
&= \int_0^x \int_0^{x_1} (x_1-t)u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1-t)^2 u(t)\,dt\,dx_1
\end{aligned}$$

Therefore,

$$\int_0^x \int_0^{x_1} (x_1-t)u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1-t)^2 u(t)\,dt\,dx_1 = \frac{1}{6}\int_0^x (x-t)^2[3+2(x-t)]u(t)\,dt.$$

From Left to Right

Combine the integrals.

$$\int_0^x \int_0^{x_1} (x_1 - t)u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1 - t)^2 u(t)\,dt\,dx_1 = \int_0^x \int_0^{x_1} [(x_1 - t) + (x_1 - t)^2]u(t)\,dt\,dx_1$$

In order to evaluate this double integral, it's necessary to switch the order of integration because $u(t)$ is not given.

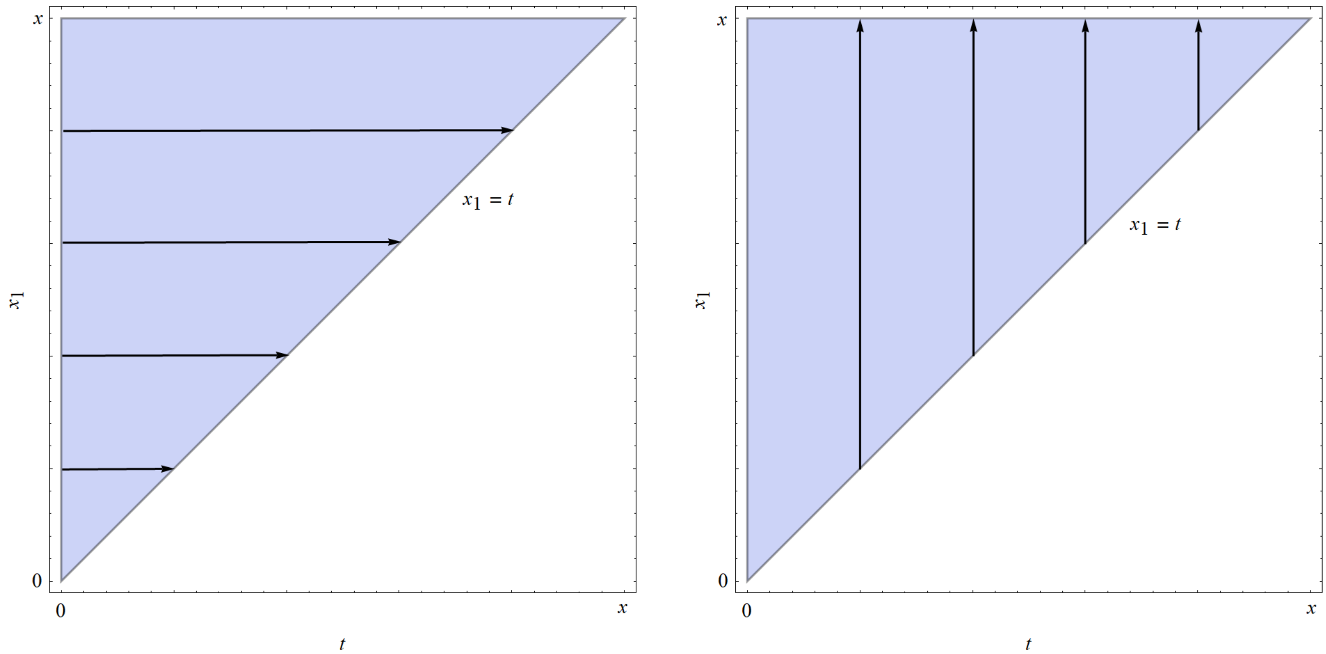

Figure 1: The current mode of integration in the tx_1-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$\begin{aligned}
\int_0^x \int_0^{x_1} [(x_1 - t) + (x_1 - t)^2]u(t)\,dt\,dx_1 &= \int_0^x \int_t^x [(x_1 - t) + (x_1 - t)^2]u(t)\,dx_1\,dt \\
&= \int_0^x \left[\frac{(x_1 - t)^2}{2} + \frac{(x_1 - t)^3}{3}\right]\Bigg|_t^x u(t)\,dt \\
&= \int_0^x \left[\frac{(x - t)^2}{2} + \frac{(x - t)^3}{3}\right] u(t)\,dt \\
&= \frac{1}{6}\int_0^x [3(x - t)^2 + 2(x - t)^3]u(t)\,dt \\
&= \frac{1}{6}\int_0^x (x - t)^2[3 + 2(x - t)]u(t)\,dt
\end{aligned}$$

Therefore,

$$\int_0^x \int_0^{x_1} (x_1 - t)u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1 - t)^2 u(t)\,dt\,dx_1 = \frac{1}{6}\int_0^x (x - t)^2[3 + 2(x - t)]u(t)\,dt.$$