Exercise 1

Solve the given ODEs:

$$y'' + 4y = 0, \; y(0) = 0, \; y'(0) = 1$$

Solution

This is a linear ODE with two initial conditions, so the Laplace transform can be used to solve the problem. The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx,$$

so the derivatives of $f(x)$ transform as follows.

$$\mathcal{L}\{f'(x)\} = sF(s) - f(0)$$
$$\mathcal{L}\{f''(x)\} = s^2F(s) - sf(0) - f'(0)$$

Take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{0\}$$

Use the fact that the operator is linear.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 0$$

Use the expressions above for the transforms of the derivatives.

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 0$$

Solve the equation for $Y(s)$.

$$(s^2 + 4)Y(s) = sy(0) + y'(0)$$
$$Y(s) = \frac{sy(0) + y'(0)}{s^2 + 4}$$

Here we use the initial conditions, $y(0) = 0$ and $y'(0) = 1$.

$$Y(s) = \frac{1}{s^2 + 4}$$

Now that we have $Y(s)$, we can take the inverse Laplace transform of it to get $y(x)$.

$$y(x) = \mathcal{L}^{-1}\{Y(s)\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\}$$

Looking in Table 1.1 on page 24, we see that this will yield a sine function. We need a 2 to be in the numerator, so place a factor of 1/2 in front.

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\}$$

Therefore,

$$y(x) = \frac{1}{2}\sin 2x.$$