Exercise 11

Find the Laplace transform of the following expressions that include convolution products:

$$\int_0^x (x-t)e^{x-t}y(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\,dx.$$

Take the Laplace transform of the provided expression.

$$\begin{aligned}\mathcal{L}\left\{\int_0^x (x-t)e^{x-t}y(t)\,dt\right\} &= \int_0^\infty e^{-sx}\int_0^x (x-t)e^{x-t}y(t)\,dt\,dx \\ &= \int_0^\infty \int_0^x e^{-sx}(x-t)e^{x-t}y(t)\,dt\,dx\end{aligned}$$

In order to evaluate the double integral, the order of integration has to be switched.

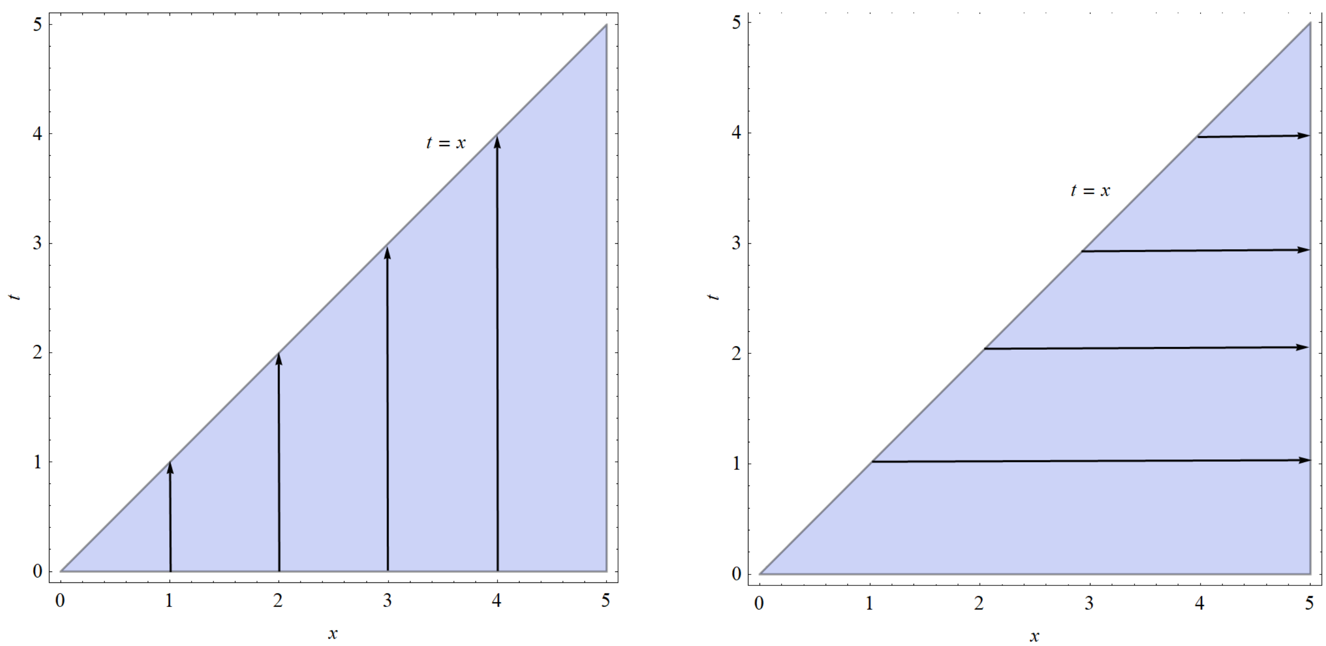

Figure 1: The current mode of integration in the xt-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the double integral.

$$\mathcal{L}\left\{\int_0^x (x-t)e^{x-t}y(t)\,dt\right\} = \int_0^\infty \int_t^\infty e^{-sx}(x-t)e^{x-t}y(t)\,dx\,dt$$

Now make the following substitution.

$$\begin{aligned} r &= x - t \quad \rightarrow \quad r + t = x \\ dr &= dx \end{aligned}$$

The double integral can then be evaluated.

$$\begin{aligned}
\mathcal{L}\left\{\int_0^x (x-t)e^{x-t}y(t)\,dt\right\} &= \int_0^\infty \int_0^\infty e^{-s(r+t)} re^r y(t)\,dr\,dt \\
&= \int_0^\infty \int_0^\infty e^{-sr}e^{-st} re^r y(t)\,dr\,dt \\
&= \left[\int_0^\infty e^{-sr} re^r\,dr\right]\left[\int_0^\infty e^{-st}y(t)\,dt\right] = \mathcal{L}\{xe^x\}\mathcal{L}\{y(x)\} \\
&= \left[\int_0^\infty \left(-\frac{\partial}{\partial s}\right) e^{-sr}e^r\,dr\right] Y(s) \\
&= -\frac{d}{ds}\left[\int_0^\infty e^{(-s+1)r}\,dr\right] Y(s) \\
&= -\frac{d}{ds}\left[\frac{1}{-s+1}e^{(-s+1)r}\bigg|_0^\infty\right] Y(s) \\
&= -\frac{d}{ds}\left(\frac{1}{s-1}\right) Y(s) \\
&= -\left[-\frac{1}{(s-1)^2}\right] Y(s)
\end{aligned}$$

Therefore,

$$\mathcal{L}\left\{\int_0^x (x-t)e^{x-t}y(t)\,dt\right\} = \frac{Y(s)}{(s-1)^2}.$$