Exercise 12

Find the Laplace transform of the following expressions that include convolution products:

$$1 + x - \int_0^x (x-t)y(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\,dx.$$

Take the Laplace transform of the provided expression.

$$\begin{aligned}
\mathcal{L}\left\{1 + x - \int_0^x (x-t)y(t)\,dt\right\} &= \int_0^\infty e^{-sx}\left[1 + x - \int_0^x (x-t)y(t)\,dt\right] dx \\
&= \int_0^\infty e^{-sx}\,dx + \int_0^\infty xe^{-sx}\,dx - \int_0^\infty e^{-sx}\int_0^x (x-t)y(t)\,dt\,dx \\
&= \frac{1}{(-s)}e^{-sx}\bigg|_0^\infty + \int_0^\infty \left(-\frac{\partial}{\partial s}\right) e^{-sx}\,dx - \int_0^\infty \int_0^x e^{-sx}(x-t)y(t)\,dt\,dx
\end{aligned}$$

In order to evaluate the double integral, the order of integration has to be switched.

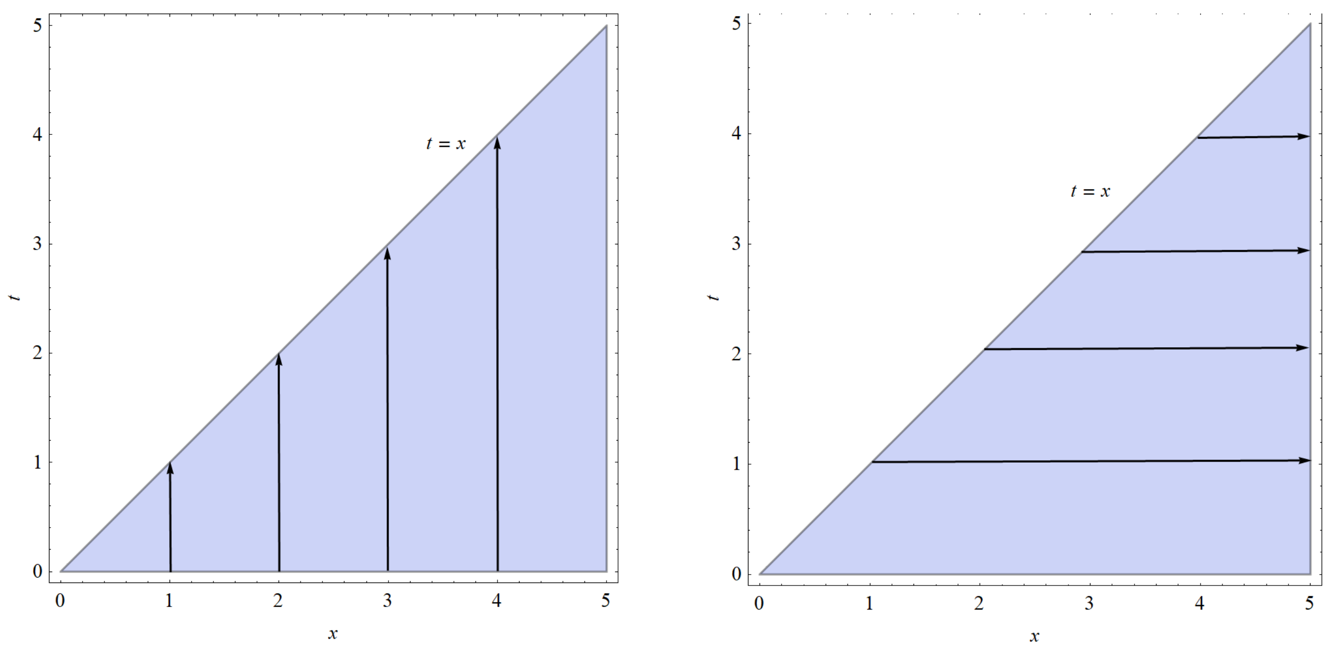

Figure 1: The current mode of integration in the xt-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the double integral.

$$\mathcal{L}\left\{1 + x - \int_0^x (x-t)y(t)\,dt\right\} = \frac{1}{s} - \frac{d}{ds}\int_0^\infty e^{-sx}\,dx - \int_0^\infty \int_t^\infty e^{-sx}(x-t)y(t)\,dx\,dt$$

Now make the following substitution.

$$\begin{aligned} r &= x - t \quad \rightarrow \quad r + t = x \\ dr &= dx \end{aligned}$$

The double integral can then be evaluated.

$$\begin{aligned} \mathcal{L}\left\{1 + x - \int_0^x (x-t)y(t)\,dt\right\} &= \frac{1}{s} - \frac{d}{ds}\left(\frac{1}{s}\right) - \int_0^\infty \int_0^\infty e^{-s(r+t)} r y(t)\,dr\,dt \\ &= \frac{1}{s} - \left(-\frac{1}{s^2}\right) - \int_0^\infty \int_0^\infty e^{-sr} e^{-st} r y(t)\,dr\,dt \\ &= \frac{1}{s} + \frac{1}{s^2} - \left[\int_0^\infty e^{-sr} r\,dr\right]\left[\int_0^\infty e^{-st} y(t)\,dt\right] = \mathcal{L}\{1\} + \mathcal{L}\{x\} - \mathcal{L}\{x\}\mathcal{L}\{y(x)\} \\ &= \frac{1}{s} + \frac{1}{s^2} - \left[\int_0^\infty \left(-\frac{\partial}{\partial s}\right) e^{-sr}\,dr\right] Y(s) \\ &= \frac{1}{s} + \frac{1}{s^2} + \frac{d}{ds}\left(\int_0^\infty e^{-sr}\,dr\right) Y(s) \\ &= \frac{1}{s} + \frac{1}{s^2} + \frac{d}{ds}\left(\frac{1}{s}\right) Y(s) \\ &= \frac{1}{s} + \frac{1}{s^2} + \left(-\frac{1}{s^2}\right) Y(s) \end{aligned}$$

Therefore,

$$\mathcal{L}\left\{1 + x - \int_0^x (x-t)y(t)\,dt\right\} = \frac{1}{s} + \frac{1}{s^2} - \frac{Y(s)}{s^2}.$$