Exercise 3

Solve the given ODEs:

$$y'' - y = -2x,\ y(0) = 0,\ y'(0) = 1$$

Solution

This is a linear ODE with two initial conditions, so the Laplace transform can be used to solve the problem. The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx,$$

so the derivatives of $f(x)$ transform as follows.

$$\mathcal{L}\{f'(x)\} = sF(s) - f(0)$$
$$\mathcal{L}\{f''(x)\} = s^2F(s) - sf(0) - f'(0)$$

Take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' - y\} = \mathcal{L}\{-2x\}$$

Use the fact that the operator is linear.

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = -2\mathcal{L}\{x\}$$

Use the expressions above for the transforms of the derivatives and use the definition on the right side.

$$s^2Y(s) - sy(0) - y'(0) - Y(s) = -2\int_0^\infty xe^{-sx}\, dx$$

Factor $Y(s)$ and solve the integral with integration by parts.

$$(s^2 - 1)Y(s) - sy(0) - y'(0) = -2\left[\frac{x}{(-s)}e^{-sx} - \frac{1}{(-s)^2}e^{-sx}\right]\Bigg|_0^\infty$$

Here we use the initial conditions, $y(0) = 0$ and $y'(0) = 1$.

$$(s^2 - 1)Y(s) - 1 = -\frac{2}{s^2}$$

Solve the equation for $Y(s)$.

$$(s^2 - 1)Y(s) = 1 - \frac{2}{s^2}$$
$$(s+1)(s-1)Y(s) = \frac{s^2 - 2}{s^2}$$
$$Y(s) = \frac{s^2 - 2}{s^2(s+1)(s-1)}$$

Use partial fraction decomposition to write the right side as a sum of simpler terms—ones that we know the inverse Laplace transforms of.

$$\frac{s^2 - 2}{s^2(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-1}$$

Multiply both sides by the LCD, $s^2(s+1)(s-1)$.

$$\begin{aligned} s^2 - 2 &= As(s+1)(s-1) + B(s+1)(s-1) + Cs^2(s-1) + Ds^2(s+1) \\ &= s^3(A+C+D) + s^2(B-C+D) + s(-A) + (-B) \end{aligned}$$

Comparing the coefficients on both sides, we obtain the following system of equations for A, B, C, and D.

$$\begin{aligned} A + C + D &= 0 \\ B - C + D &= 1 \\ -A &= 0 \\ -B &= -2 \end{aligned}$$

The results are $A = 0$, $B = 2$, $C = 1/2$, and $D = -1/2$. Thus,

$$Y(s) = \frac{2}{s^2} + \frac{1/2}{s+1} - \frac{1/2}{s-1}.$$

Now that we have $Y(s)$, we can obtain $y(x)$ by taking the inverse Laplace transform of it.

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{2}{s^2} + \frac{1/2}{s+1} - \frac{1/2}{s-1}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\ &= 2x + \frac{1}{2}e^{-x} - \frac{1}{2}e^{x} \\ &= 2x - \frac{e^x - e^{-x}}{2} \end{aligned}$$

Therefore,

$$y(x) = 2x - \sinh x.$$